

2025 VCE Mathematical Methods Examination 2 Solutions *with*

TI-Nspire™ CX II CAS
graphing calculator



Teachers Teaching with Technology™
Professional Development from Texas Instruments

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

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www.education.ti.com/aus/vic



Section A – Multiple-choice questions

Instructions

- Answer **all** questions in pencil on your Multiple-Choice Answer Sheet.
- Choose the response that is **correct** for the question.
- A correct answer scores 1; an incorrect answer scores 0.
- Marks will **not** be deducted for incorrect answers.
- No marks will be given if more than one answer is completed for any question.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

A function that has a range of $[6, 12]$ is

A. $f: R \rightarrow R, f(x) = 6 + 3 \cos(9x)$

B. $f: R \rightarrow R, f(x) = 6 + 6 \cos(3x)$

C. $f: R \rightarrow R, f(x) = 9 - 3 \cos(6x)$

D. $f: R \rightarrow R, f(x) = 9 - 6 \cos(3x)$

$$f(x) = A + B \cos(Cx)$$

$$\text{Min} = 9 - 3 = 6$$

$$\text{Max} = 9 + 3 = 12$$

Question 2

All asymptotes of the graph of $y = 2 \tan\left(\pi\left(x + \frac{1}{2}\right)\right)$ are given by

A. $x = k, k \in Z$

B. $x = 2k, k \in Z$

C. $x = 2k + 1, k \in Z$

D. $x = \frac{4k+1}{2}, k \in Z$

For graphs of $y = a \tan(b(x+c))$,

$$\text{VA at } x = \frac{(2k+1)\pi}{2b} - c, k \in Z.$$

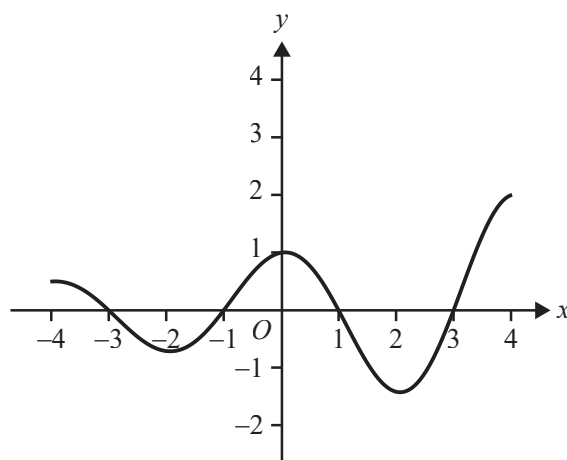
$$\text{For } a=2, b=\pi, c=\frac{1}{2},$$

$$\text{VA at } x = \frac{(2k+1)\pi}{2(\pi)} - \frac{1}{2} = k.$$

$$\frac{(2 \cdot k+1) \cdot \pi}{2 \cdot b} - c | b=\pi \text{ and } c=\frac{1}{2} \quad k$$

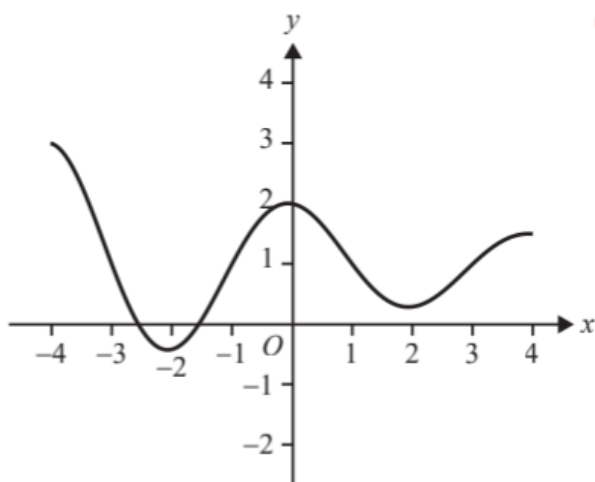
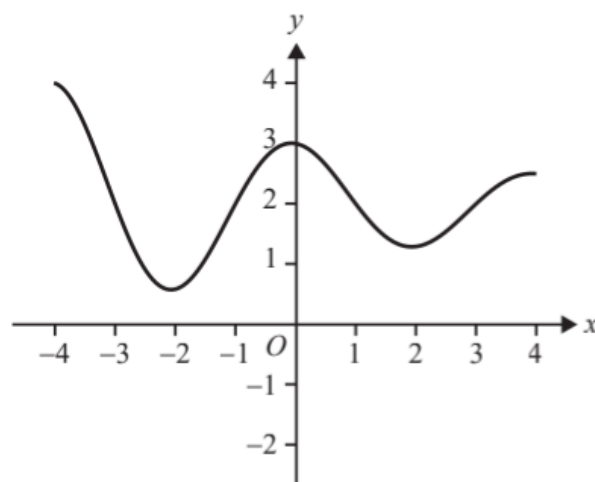
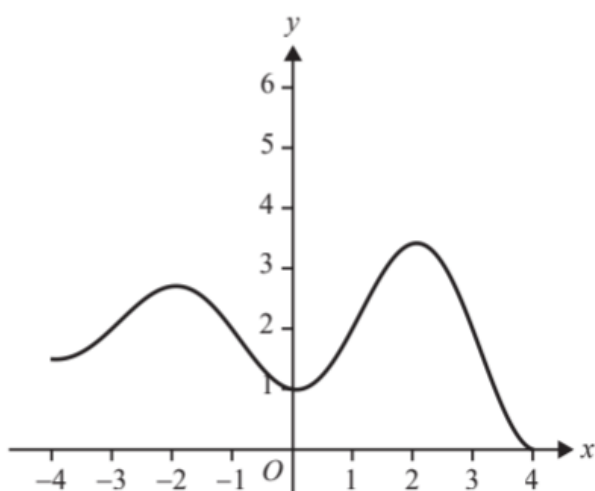
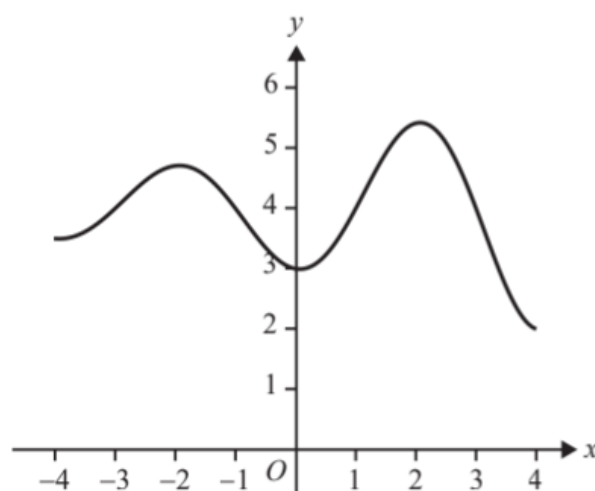
Question 3

The graph of $y = f(x)$ is shown below.



Which one of the following options best represents the graph of $y = f(-x) + 2$?

Reflected in the y -axis,
Translated 2 units up.

A.**B.****C.****D.**

Question 4

Consider the system of equations below containing the parameter k , where $k \in \mathbb{R}$.

$$\begin{aligned} kx + 3y &= k^2 \\ 2x + (2k+1)y &= 6 - 2k \end{aligned}$$

$$\text{solve } \left(\frac{k}{3} = \frac{2}{2 \cdot k+1} \text{ and } \frac{k^2}{3} \neq \frac{6-2 \cdot k}{2 \cdot k+1}, k \right)$$

$$k = -2$$

Find the value(s) of k for which this system has no real solutions

- A.** $k = -2$ only
- B.** $k = \frac{3}{2}$ only
- C.** $k = -2$ or $\frac{3}{2}$
- D.** $k \in \mathbb{R} \setminus \left\{ -2, \frac{3}{2} \right\}$

For no real solutions, the lines associated with the two linear equations must have the same gradient but different y -intercepts.

System of equations	Same gradients if:	Different y -intercepts if:
$ax + by = c$ $dx + ey = f$	$\frac{a}{b} = \frac{d}{e}$	$\frac{c}{b} \neq \frac{f}{e}$
In this question ...	Same gradients if: $\frac{k}{3} = \frac{2}{2k+1}$	Different y -intercepts if: $\frac{k^2}{3} \neq \frac{6-2k}{2k+1}$

Question 5

Which of the following sets represents a function that has an inverse function?

- A.** $\{(1, 3), (2, 0), (2, 1)\}$
- B.** $\{(-1, 3), (2, 2), (3, 1)\}$
- C.** $\{(-1, 3), (0, 1), (1, 3)\}$
- D.** $\{(1, 0), (2, 3), (1, 3)\}$

Functions must be one-to-one to have an inverse function.

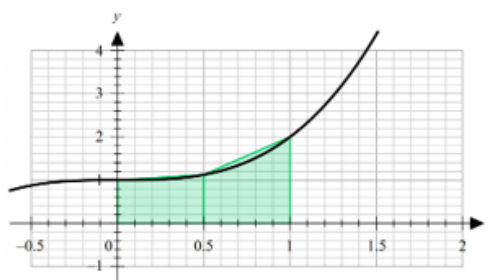
Question 6

The trapezium rule is used, with two trapeziums, to estimate the area bounded by the graph of $y = f(x)$, the x -axis and the lines $x = 0$ and $x = 1$.

For which function will the trapezium rule estimate be **larger** than the exact area?

- A.** $f(x) = 3 - e^x$
- B.** $f(x) = x^3 + 1$
- C.** $f(x) = 3 \sin(x) + 1$
- D.** $f(x) = \log_e(x + 3)$

The only graph which is concave up and increasing between $x = 0$ and $x = 1$ is the graph of $f(x) = x^3 + 1$.



Question 7

Consider the algorithm below.

```

n ← 17
k ← 5
while n > k
    n ← n - k
    print n
end while

```

In order, the values printed by the algorithm are

- A. 12
- B. 12, 7
- C. 12, 7, 2**
- D. 12, 7, 2, -3

$n = 17, k = 5, n > k$
 $n = 12, k = 5, \text{print } 12, n > k$
 $n = 7, k = 5, \text{print } 7, n > k$
 $n = 2, k = 5, \text{print } 2, n < k \text{ end while}$
 The printed values are 12, 7 and 2.

Question 8

A random sample of n Victorian households is taken to estimate the proportion of all Victorian households that have vegetable gardens. The approximate 95% confidence interval calculated using this sample is (0.248, 0.552), correct to three decimal places.

The number of households, n , in the sample is

- A. 10
- B. 28
- C. 40**
- D. 49

$$\hat{p} \approx \frac{0.248 + 0.552}{2} = 0.4$$

$$1.96 \sqrt{\frac{0.4 \times 0.6}{n}} \approx 0.152, n \approx 39.905 \dots$$

$$n = 40$$

$$\text{phat} := \frac{0.248 + 0.552}{2} \quad 0.4$$

$$\text{solve}\left(\text{phat} + 1.96 \cdot \sqrt{\frac{\text{phat} \cdot (1 - \text{phat})}{n}} = 0.552, n\right)$$

$$n = 39.9058$$

Question 9

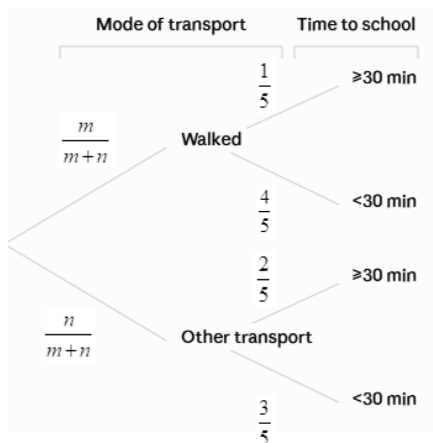
One day, at a particular school, m students walked to school and the remaining n students travelled to school using a different form of transport.

Of the m students who walked, 20% took at least 30 minutes to get to school.

Of the n students who used a different form of transport, 40% took at least 30 minutes to get to school.

Given that a randomly selected student took at least 30 minutes to get to school, the probability that they walked to school is given by

- A. $\frac{m}{m+2n}$
- B. $\frac{2n}{m+2n}$
- C. $\frac{m}{5(m+n)}$
- D. $\frac{1}{3}$



$$\begin{aligned}\Pr(\text{walk} | \geq 30 \text{ minutes}) &= \frac{\Pr(\text{walk} \cap \geq 30 \text{ minutes})}{\Pr(\geq 30 \text{ minutes})} \\ &= \frac{\frac{m}{m+n} \times \frac{1}{5}}{\frac{m}{m+n} \times \frac{1}{5} + \frac{n}{m+n} \times \frac{2}{5}} \\ &= \frac{m}{m+2n}\end{aligned}$$

$$\frac{\frac{m}{m+n} \cdot \frac{1}{5}}{\frac{m}{m+n} \cdot \frac{1}{5} + \frac{n}{m+n} \cdot \frac{2}{5}} = \frac{m}{m+2n}$$

Question 10

Consider $f: R \rightarrow R$, $f(x) = 2x^2 + x - 1$ and $g: R \rightarrow R$, $g(x) = \sin(x)$.

The inequality $(f \circ g)(x) > 0$ is satisfied when

- A. $\sin(x) \leq -1$
- B. $-1 < \sin(x) < 0$
- C. $\frac{1}{2} < \sin(x) \leq 1$
- D. $0 < \sin(x) < \frac{1}{2}$

$$(f \circ g)(x) = 2(\sin(x))^2 + \sin(x) - 1$$

$$\text{Let } a = \sin(x) \text{ and } y = (f \circ g)(x).$$

$$y = 2a^2 + a - 1 = (2a - 1)(a + 1)$$

$$= 0 \text{ when } a = -1, \frac{1}{2}$$

As the graph is a parabola, $y > 0$ for $a < -1 \cup a > \frac{1}{2}$.

So $(f \circ g)(x) > 0$ if $\sin(x) < -1 \cup \sin(x) > \frac{1}{2}$

However, $\sin(x)$ only exists between -1 and 1 inclusive,

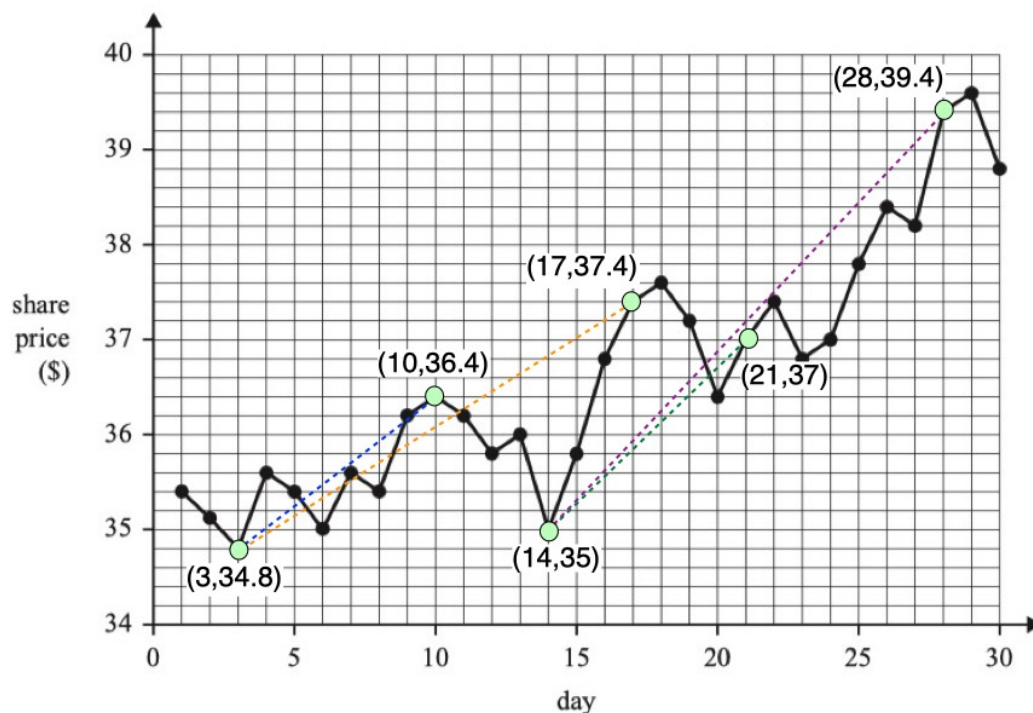
so $\frac{1}{2} < \sin(x) \leq 1$.

$$\text{solve}(2 \cdot a^2 + a - 1 > 0, a) \quad a < -1 \text{ or } a > \frac{1}{2}$$

$$\text{solve}(2 \cdot a^2 + a - 1 > 0, a) | -1 < a < 1 \quad \frac{1}{2} < a < 1$$

Question 11

The chart below shows the daily price of a stock market share over a 30-day period.



Over which of the following time intervals did the daily price undergo the greatest average rate of change?

- A. day 3 to day 10
- B. day 3 to day 17
- C. day 14 to day 21
- D. day 14 to day 28**

$$\text{day 3 to day 10: Ave. ROC} = \frac{36.4 - 34.8}{10 - 3} = \frac{1.6}{7} \approx 0.23$$

$$\text{day 3 to day 17: Ave. ROC} = \frac{37.4 - 34.8}{17 - 3} = \frac{2.6}{14} \approx 0.19$$

$$\text{day 14 to day 21: Ave. ROC} = \frac{37 - 35}{21 - 14} = \frac{2}{7} \approx 0.29$$

$$\text{day 14 to day 28: Ave. ROC} = \frac{39.4 - 35}{28 - 14} = \frac{4.4}{14} \approx 0.31$$

Question 12

For a normal random variable X , it is known that $\Pr(X > 200) = 0.325$ and $\Pr(180 < X < 200) = 0.589$

The mean and standard deviation of X are closest to

- A. 190 and 10
- B. 190 and 11
- C. 195 and 10
- D. 195 and 11**

$$\Pr(X > 200) = 0.325$$

$$\therefore \Pr(Z < z_1) = 1 - 0.325 = 0.675$$

$$\therefore z_1 = \frac{200 - \mu}{\sigma} = 0.453...$$

$$\Pr(X < 180) = \Pr(Z < z_2) = 0.675 - 0.589 = 0.086$$

$$\therefore z_2 = \frac{180 - \mu}{\sigma} = -1.365...$$

$$\mu = 195, \sigma = 11$$

$$\text{invNorm}(0.675, 0, 1) \quad 0.4537622$$

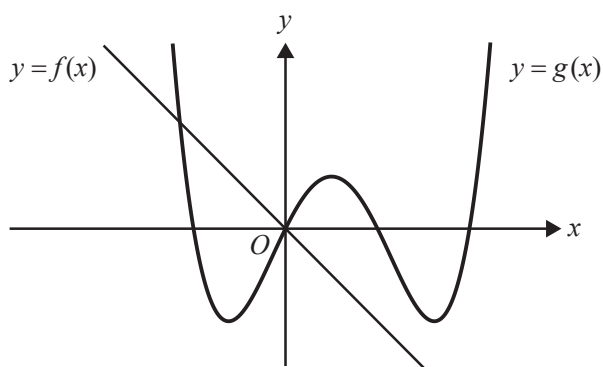
$$\text{invNorm}(0.086, 0, 1) \quad -1.365806$$

$$\text{solve} \left(\begin{cases} \frac{200-x}{y} = 0.4537621933589 \\ \frac{180-x}{y} = -1.3658055634504 \end{cases}, \{x, y\} \right)$$

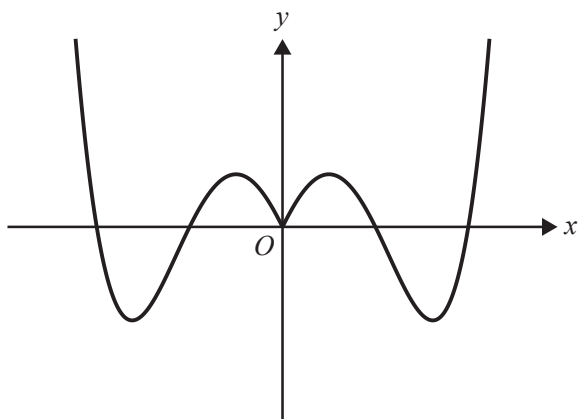
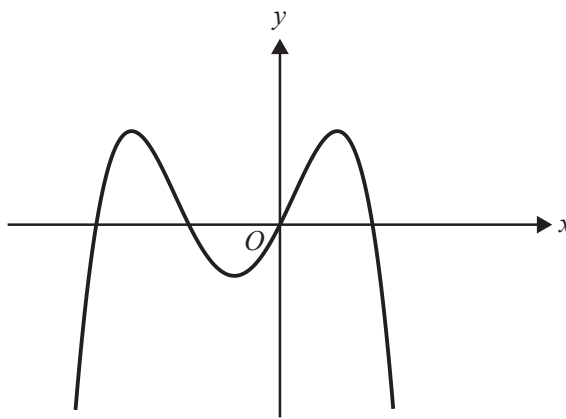
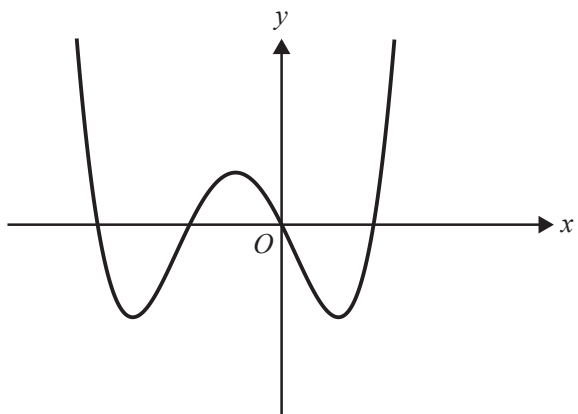
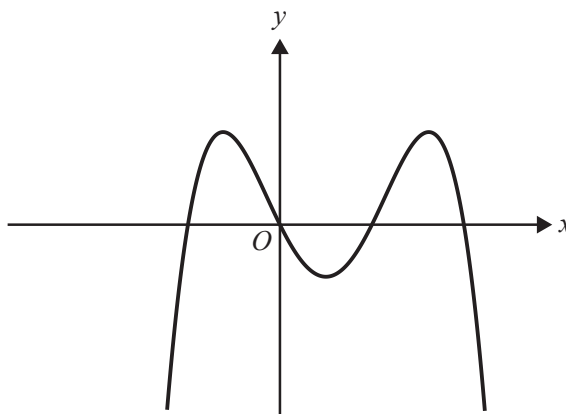
$$x=195.0124 \text{ and } y=10.99162$$

Question 13

The graphs of $y = f(x)$ and $y = g(x)$ are sketched on the same set of axes below.



Which of the following could be the graph of $y = (g \circ f)(x)$?

A.**B.****C.****D.**

The graph of $g(x)$ suggests a positive quartic, and $f(x)$ suggests a negative linear function. A positive quartic with x replaced with a negative linear function remains a quartic, but a reflection in the y -axis.

The only graph which represents this is option C.

Question 14

Let f be the probability density function for a continuous random variable X , where

$$f(x) = \begin{cases} k \sin(x) & 0 \leq x < \frac{\pi}{4} \\ k \cos(x) & \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

and k is a positive real number.

The value of k is

- A. $\frac{1}{\sqrt{2}}$
B. $\frac{1}{2 - \sqrt{2}}$
 C. $\sqrt{2} + 2$
 D. $2 - \sqrt{2}$

$$f(x) := \begin{cases} k \cdot \sin(x), & 0 \leq x \leq \frac{\pi}{4} \\ k \cdot \cos(x), & \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ 0, & \text{Else} \end{cases} \quad \text{Done}$$

$$\text{solve} \left(\int_0^{\frac{\pi}{2}} f(x) \, dx = 1, k \right) \quad k = \frac{\sqrt{2} + 2}{2}$$

$$\frac{\sqrt{2} + 2}{2} = \frac{1}{2 - \sqrt{2}} \quad \text{true}$$

Question 15

The graph of $y = g(x)$ passes through the point $(1, 3)$.

The graph of $y = 1 - g(2x - 3)$ must pass through the point

- A. $(-1, -2)$
B. $(2, -2)$
 C. $(-1, 2)$
 D. $(2, 2)$

$$y = 1 - g(2x - 3) = -g\left(2\left(x - \frac{3}{2}\right)\right) + 1$$

Reflect $(1, 3)$ in the x -axis gives $(1, -3)$.

Dilate by a factor of $\frac{1}{2}$ from the y -axis gives $(1/2, -3)$.

Translate 1 unit up and $3/2$ units to the right gives $(2, -2)$.

Question 16

Consider the function $h(x) = a \log_e(bx)$, where $a, b \in \mathbb{R} \setminus \{0\}$.

Given that its derivative $h'(x)$ has range $(0, \infty)$, which of the following **must** be true?

- A. $a > 0$ only
 B. $a > 0$ and $b < 0$
 C. $a > 0$ and $b > 0$
D. $ab > 0$

If $\text{ran } h'(x) = (0, \infty)$, then $h(x)$ is strictly increasing.

(i.e. $h'(x) > 0$ for $\text{dom } h(x)$).

Consider the four cases where $a = \pm 1$ and $b = \pm 1$.

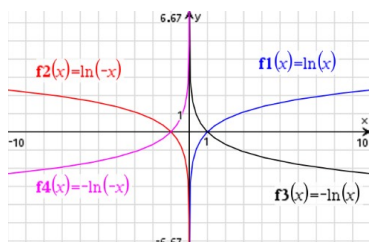
Case 1: If $a = 1$ and $b = 1$, $h(x) = \ln(x)$ (strictly increasing)

Case 2: If $a = 1$ and $b = -1$, $h(x) = \ln(-x)$ (strictly decreasing)

Case 3: If $a = -1$ and $b = 1$, $h(x) = -\ln(x)$ (strictly decreasing)

Case 4: If $a = -1$ and $b = -1$, $h(x) = -\ln(-x)$ (strictly increasing)

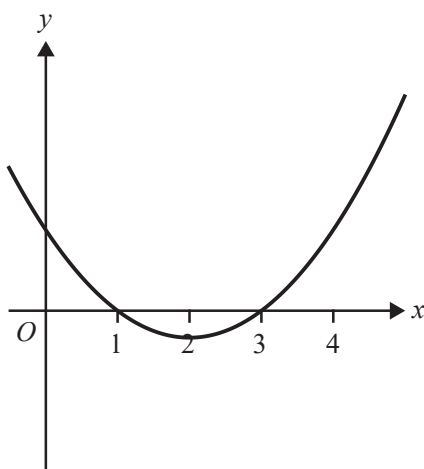
Conclusion: As only Case 1 ($a > 0, b > 0$) and Case 4 ($a < 0, b < 0$) will result in increasing functions, the only true statement is $ab > 0$.



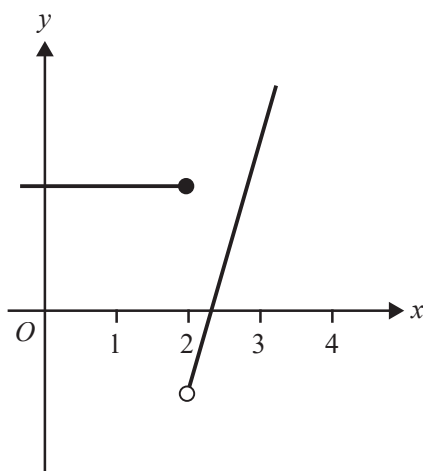
Question 17

Given that $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies $\int_1^2 f(x) dx > \int_1^3 f(x) dx$, the graph of $y = f(x)$ could be

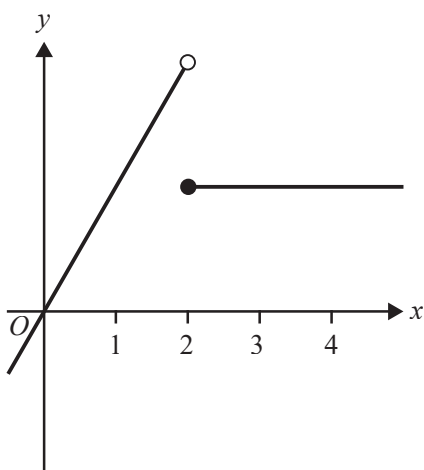
A.



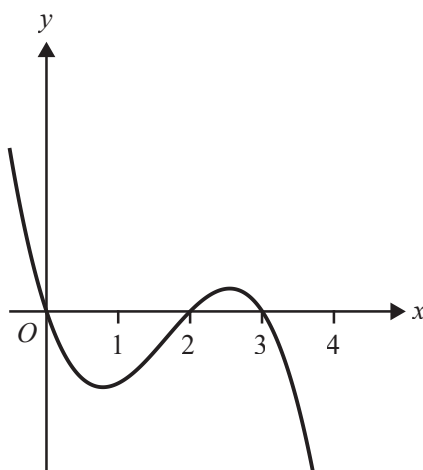
B.



C.



D.



$$\int_1^2 f(x) dx > \int_1^3 f(x) dx$$

$$\int_1^2 f(x) dx - \int_1^3 f(x) dx > 0$$

$$\int_1^2 f(x) dx - \left(\int_1^2 f(x) dx + \int_2^3 f(x) dx \right) > 0$$

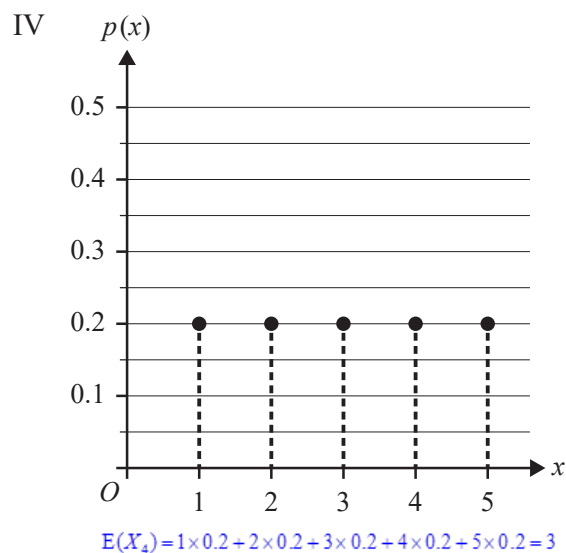
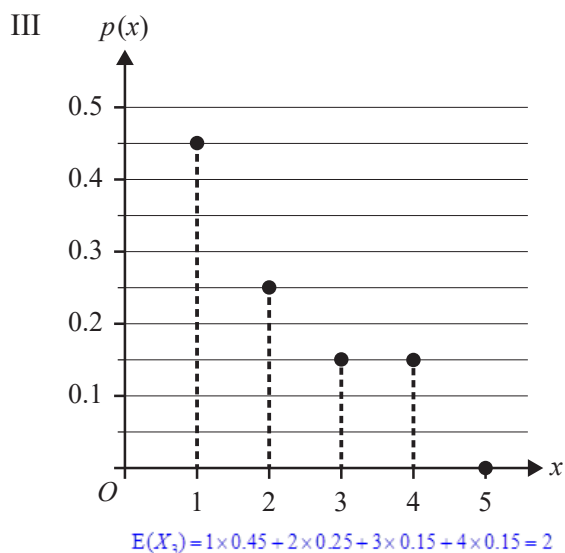
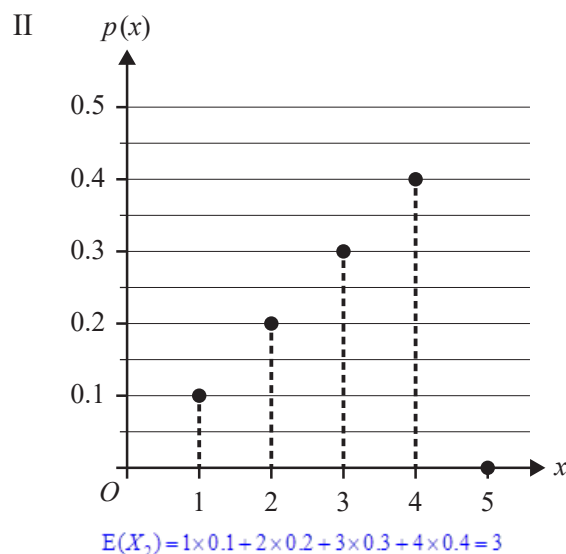
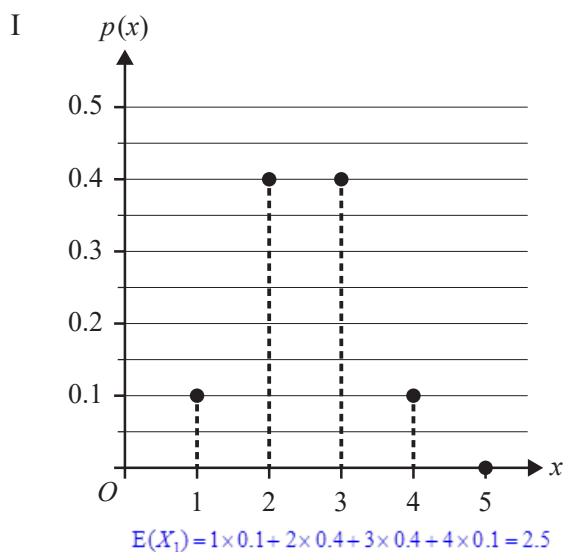
$$\int_2^3 f(x) dx > 0$$

$$\int_2^3 f(x) dx < 0$$

The only graph where this condition applies is in Option A.

Question 18

Consider the following graphs, which represent probability mass functions.



Which pair of these probability mass functions has the same mean?

- A. I and II
- B. I and IV
- C. II and III
- D. II and IV**

Question 19

Let A be a point on the line $y = x + c$ and B be a point on the curve $y = \log_e(x - 1)$.

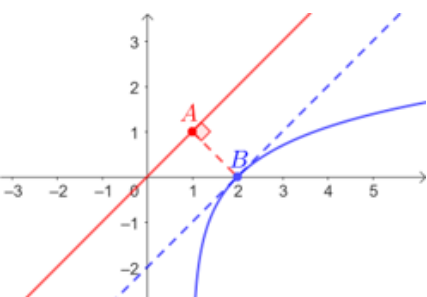
If A and B are placed such that the line segment AB has the minimum possible length, and this length is $\sqrt{2}$, the value of c must be

- A. $\sqrt{2} - 2$
 B. $\sqrt{2}$
 C. 1
D. 0

Find x -coordinate of B , where tangent to curve is parallel to line $y = x + c$.

Define the distance AB .

Find the x -coordinate of the point on $y = x + c$ associated with the minimum distance AB .



The solution $c = -4$ implies the line is $y = x - 4$, which intersects the log curve (so minimum distance = 0), so $c = 0$ is the only possible value of c .

$f(x) := x + c$	Done
$g(x) := \ln(x - 1)$	Done
Δ solve $\left(\frac{d}{dx}(g(x)) = 1, x\right)$	$x = 2$
$g(2)$	0
$d(x) := \sqrt{(x - 2)^2 + (f(x) - 0)^2}$	Done
solve $\left(\frac{d}{dx}(d(x)) = 0, x\right)$	$x = \frac{-(c - 2)}{2}$
solve $\left(d\left(\frac{-(c - 2)}{2}\right) = \sqrt{2}, c\right)$	$c = -4$ or $c = 0$

Question 20

Let $a > 1$, and consider the functions f and g defined below.

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = a^x$$

$$g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = a^{2x+2}$$

Which one of the following sequences of transformations, when applied to $f(x)$, does **not** produce $g(x)$?

- A. dilation by a factor of $\frac{1}{2}$ from the y -axis, then translation by 1 unit in the negative direction of the x -axis
- B. dilation by a factor of $\frac{1}{2}$ from the y -axis, then dilation by a factor of a^2 from the x -axis
- C.** dilation by a factor of a from the x -axis, then dilation by a factor of $\frac{1}{2}$ from the y -axis, then translation by 1 unit in the positive direction of the x -axis
- D. dilation by a factor of a^3 from the x -axis, then translation by 1 unit in the positive direction of the x -axis, then dilation by a factor of $\frac{1}{2}$ from the y -axis

	$f(x) := a^x$	Done
A	$\Delta f(2 \cdot (x + 1))$	$a^{2 \cdot x + 2}$
B	$a^2 \cdot f(2 \cdot x)$	$a^{2 \cdot x + 2}$
C	$\Delta a \cdot f(2 \cdot (x - 1))$	$a^{2 \cdot x - 1}$
D	$\Delta a^3 \cdot f(2 \cdot x - 1)$	$a^{2 \cdot x + 2}$

Section B

Instructions

- Answer **all** questions in the spaces provided.
- Write your responses in English.
- In questions where a numerical answer is required, an **exact** value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (13 marks)

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $g(x) = 4x^3 - 3x^4$.

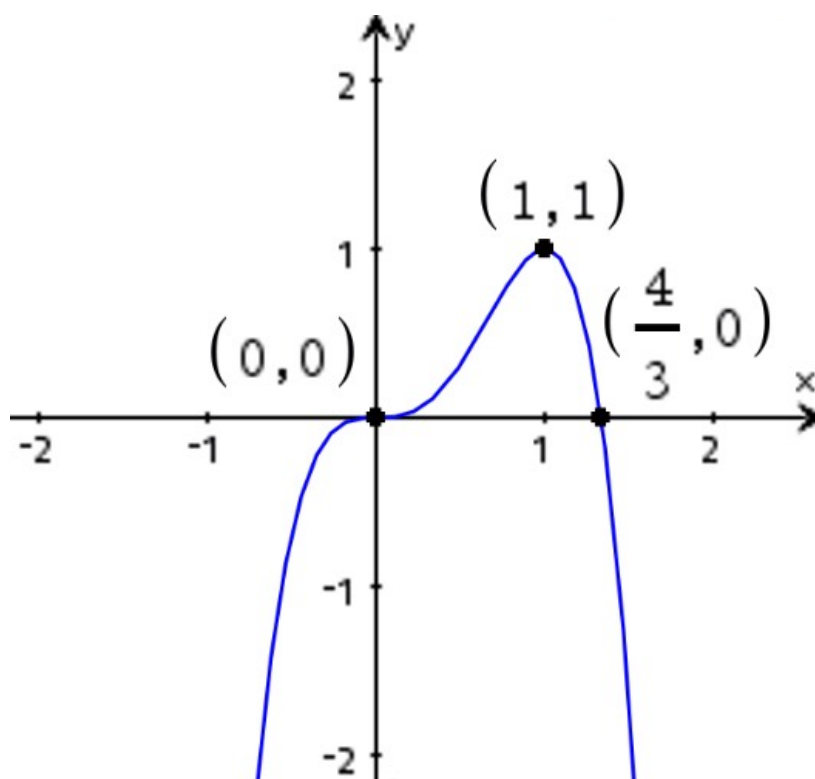
- a. Find the coordinates of both stationary points of g .

2 marks

From graph $(0, 0)$ and $(1, 1)$

- b. Sketch the graph of $y = g(x)$ on the axes below, labelling the stationary points and axial intercepts with their coordinates.

2 marks



- c. Complete the following gradient table with appropriate values of x and $g'(x)$ to show that g has a stationary point of inflection.

2 marks

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
$g'(x)$	$\frac{9}{2}$	0	$\frac{3}{2}$

$$\frac{d}{dx}(g(x))|_{x=\left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}} = \left\{\frac{9}{2}, 0, \frac{3}{2}\right\}$$

- d. Find the average value of g between $x=0$ and $x=2$.

2 marks

$$\text{Average value} = \frac{1}{2-0} \int_0^2 g(x) dx = -\frac{8}{5}$$

$$\frac{1}{2} \int_0^2 g(x) dx = \frac{-8}{5}$$

- e. Let h be the result after applying a sequence of transformations to g , such that h has a stationary point of inflection at $(1, 0)$ and a local maximum at $(-1, 1)$.

Write down a possible sequence of three transformations to map from g to h .

3 marks

1. Reflect in the y-axis

1. Reflect in the y-axis

2. Dilate by factor 2 from the y-axis

OR

2. Translate 0.5 units right

3. Translate 1 unit right

3. Dilate by factor 2 from the y-axis

OR

OR

1. Dilate by factor 2 from the y-axis

1. Translate 0.5 units left

2. Translate 1 unit left

OR

2. Dilate by factor 2 from the y-axis

3. Reflect in the y-axis

3. Reflect in the y-axis

- f. Let $X \sim \text{Bi}(4, p)$ be a binomial random variable.

Show that $\Pr(X \geq 3) = g(p)$ for all $p \in [0, 1]$.

2 marks

$$\Pr(X \geq 3)$$

$$= \Pr(X = 3) + \Pr(X = 4)$$

$$= {}^4C_3 p^3 (1-p)^1 + {}^4C_4 p^4 (1-p)^0$$

$$= 4p^3(1-p) + p^4$$

$$= 4p^3 - 3p^3$$

$$= g(p)$$

Question 2 (14 marks)

Let

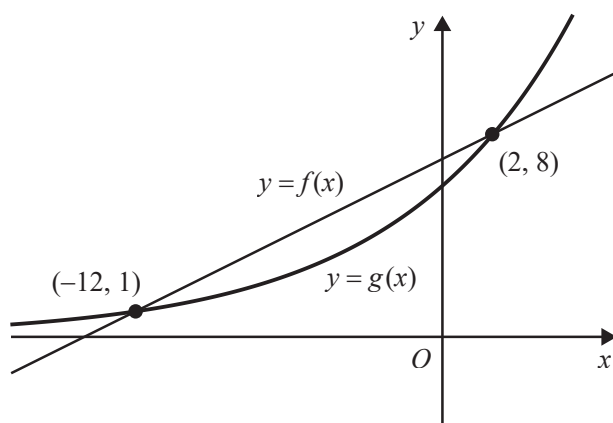
$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \frac{x}{2} + 7$$

and

$$g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = Ae^{kx}$$

where $A, k \in \mathbb{R}$.

The graphs of $y = f(x)$ and $y = g(x)$ intersect at the points $(-12, 1)$ and $(2, 8)$, as shown below.



- a. Write down two simultaneous equations in terms of A and k .

Solve them, using algebra, to show that $A = 2^{\frac{18}{7}}$ and $k = \frac{3}{14} \log_e(2)$.

3 marks

$$1 = Ae^{-12k} \quad \dots(1)$$

$$8 = Ae^{2k} \quad \dots(2)$$

Divide (2) by (1)

$$8 = e^{14k}$$

$$14k = \log_e(8)$$

$$14k = 3 \log_e(2)$$

$$k = \frac{3}{14} \log_e(2)$$

sub into (1)

$$1 = Ae^{-12\left(\frac{3}{14} \log_e(2)\right)}$$

$$1 = Ae^{-\frac{18}{7} \log_e(2)}$$

$$1 = A \times 2^{-\frac{18}{7}}$$

$$\therefore A = 2^{\frac{18}{7}}$$

- b. Find the value of b , where $b \in \mathbb{R}$, such that $g(x)$ can be expressed in the form $g(x) = A \times 2^{bx}$.

1 mark

$$b = \frac{3}{14}$$

$$\frac{3}{a \cdot e^{14}} \cdot \ln(2) \cdot x$$

$$\frac{3 \cdot x}{a \cdot 2^{14}}$$

- c. Use a definite integral to evaluate the area bounded by the graphs of $y = f(x)$ and $y = g(x)$, where $x \in [-12, 2]$.

Give the area correct to **two decimal places**.

2 marks

$$\int_{-12}^2 (f(x) - g(x)) dx$$

$$= 15.87$$

$$f(x) := \frac{x}{2} + 7$$

Done

$$g(x) := 2^{\frac{18}{7} + \frac{3 \cdot x}{14}}$$

Done

$$\int_{-12}^2 (f(x) - g(x)) dx$$

15.87196

- d. Let $h(x) = f(x) - g(x)$.

- i. Write down an expression for the derivative of $h(x)$.

1 mark

$$h'(x) = \frac{1}{2} + \frac{6 \times 2^{\frac{3x}{14} + \frac{4}{7}} \ln(2)}{7}$$

$$\frac{d}{dx}(f(x) - g(x))$$

$$\frac{1}{2} - \frac{6 \cdot 2^{\frac{3 \cdot x}{14} + \frac{4}{7}} \cdot \ln(2)}{7}$$

- ii. Find the maximum value of $h(x)$, where $x \in [-12, 2]$.

Give your answer correct to **two decimal places**.

1 mark

$$\text{Max } h(x) = 1.72 \text{ at } x = -3.83$$

$$\text{solve}\left(\frac{d}{dx}(f(x) - g(x)) = 0, x\right)$$

 $x = -3.827926$

$$f(x) - g(x) |_{x = -3.8279256266874}$$

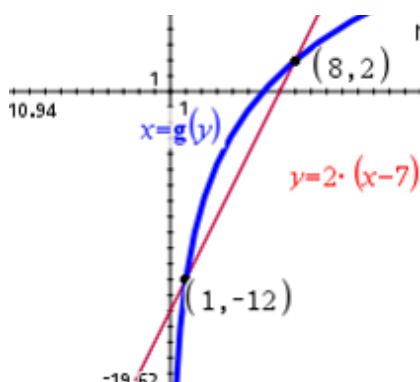
1.719749

- e. Let g^{-1} be the inverse of g .

Find the points where the graph of $y = g^{-1}(x)$ intersects with the graph of $y = 2(x - 7)$.

2 marks

Use Relation (via Menu > Graph Entry/Edit)



(1, -12) and (8, 2)

$$\text{solve}(x = g(y) \text{ and } y = 2 \cdot (x - 7), x, y)$$

 $x = 1. \text{ and } y = -12. \text{ or } x = 8. \text{ and } y = 2.$

Question 2 continues on the next page.

f. Let F be an anti-derivative of f that passes through $(0, c)$, where $c \in \mathbb{R}$.

- i. Show that it is **not** possible for the graph of $y = F(x)$ to pass through both $(-12, 1)$ and $(2, 8)$.

2 marks

$F(0) = c \Rightarrow c' = c$	_____	$\int f(x) \, dx$ $\frac{x^2}{4} + 7x$
$F(x) = \int f(x) dx$	_____	
$= \frac{1}{4}x^2 + 7x + c$	_____	$\int f(x) \, dx _{x=\{-12, 2\}}$ $\{-48, 15\}$
$F(12) = 1 = -48 + c$	_____	
$\therefore c' = 49$	_____	
OR	_____	
$F(2) = 8 = 15 + c$	_____	
$\therefore c' = -7$	_____	

However c cannot be two values _____

$\therefore F(x)$ cannot pass through both $(-12, 1)$ and $(2, 8)$ _____

- ii. The graph of $y = F(x)$ can be dilated by a factor of m from the x -axis such that its image passes through both $(-12, 1)$ and $(2, 8)$.

Find the values of m and c .

2 marks

$F(x) = \int f(x) dx$	_____	$\text{solve} \left(\begin{cases} m \cdot \left(\int f(x) \, dx + c \right) = 1 & x = -12 \\ m \cdot \left(\int f(x) \, dx + c \right) = 8 & x = 2 \end{cases}, \{m, c\} \right)$ $c = 57 \text{ and } m = \frac{1}{9}$
$= \frac{1}{4}x^2 + 7x + c$	_____	
$mF(-12) = 1$	_____	
$mF(2) = 8$	_____	
$\therefore c = 57 \text{ and } m = \frac{1}{9}$	_____	

Question 3 (14 marks)

The time taken for a driver to travel to work each day, in minutes, is modelled by a continuous random variable T with probability density function

$$f(t) = \begin{cases} \frac{1}{1215000} (t-29)(59-t)^3 & 29 \leq t \leq 59 \\ 0 & \text{otherwise} \end{cases}$$

- a. i. Find the mean time taken, in minutes, for the driver to travel to work each day. 1 mark

$$\begin{aligned} \text{Mean} &= \int_{29}^{59} t \times f(t) dt \\ &= 39 \end{aligned}$$

$$f(t) = \begin{cases} \frac{1}{1215000} \cdot (t-29) \cdot (59-t)^3, & 29 \leq t \leq 59 \\ 0, & \text{Else} \end{cases}$$

Done

$$\int_{29}^{59} (t \cdot f(t)) dt = 39$$

- ii. Find the standard deviation of the time taken, in minutes, for the driver to travel to work each day. 2 marks

$$\begin{aligned} \text{Standard deviation} &= \sqrt{\int_{29}^{59} t^2 \times f(t) dt - \left(\int_{29}^{59} t \times f(t) dt \right)^2} \\ &= \frac{10\sqrt{14}}{7} \end{aligned}$$

$$\sqrt{\int_{29}^{59} (t^2 \cdot f(t)) dt - \left(\int_{29}^{59} (t \cdot f(t)) dt \right)^2} = \frac{10 \cdot \sqrt{14}}{7}$$

- b. The driver allows k minutes to travel to work each day. If the journey takes longer than k minutes, the driver will be late. Whether the driver is late on a particular day is independent of whether they are late on any other day.

- i. If $k = 47$, write a definite integral to show that the probability of the driver being late is 0.08704

1 mark

$$\int_{47}^{59} f(t) dt = 0.08704$$

- ii. If $k = 47$, find the probability that the driver will be late on at least one day in a five-day working week.

Give your answer correct to **four decimal places**.

2 marks

Let X be the no. of days the driver will be late in a five-day working week.

$$X \sim \text{Bi}(5, 0.08704)$$

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$

$$= 1 - (1 - 0.08704)^5$$

$$= 0.3658$$

$$1 - (1 - 0.08704)^5$$

$$0.3658$$

$$\text{binomCdf}(5, 0.08704, 1, 5)$$

$$0.3658$$

- iii. For $k = 47$, let $\hat{P}$ be the proportion of days the driver is late in any five-day working week. Find $\Pr(0.4 \leq \hat{P} \leq 0.6)$ correct to **four decimal places**.

2 marks

$$\hat{p} = \frac{x}{5}$$

$$\text{binomCdf}(5, 0.08704, 2, 3)$$

$$0.0631$$

$$\Pr(0.4 \leq \hat{P} \leq 0.6)$$

$$= \Pr(2 \leq X \leq 3)$$

$$= 0.0631$$

- iv. Find the **integer** k such that the probability, correct to one decimal place, of the driver being late at least once in any five-day working week is 0.2

2 marks

Let Y be the no. of days the driver will be late in a five-day working week (p unknown).

$$Y \sim \text{Bi}(5, p)$$

$$\Pr(Y \geq 1) = 1 - \Pr(Y = 0)$$

$$= 1 - (1 - p)^5$$

$$= 0.2$$

$$\therefore 1 - \left(1 - \int_k^{59} f(t) dt\right)^5 = 0.2$$

$$\therefore k = 49$$

$$\text{solve}\left(1 - \left(1 - \int_k^{59} f(x) dx\right)^5 = 0.2, k\right) \quad k=49.1$$

c. At a given traffic light, the wait time is modelled by a normal distribution with a mean of 2.5 minutes and a standard deviation of σ minutes.

i. If $\sigma = 0.6$, find the probability that the wait time will be less than 3.5 minutes.

Give your answer correct to two decimal places. 1 mark

0.95

normCdf($-\infty, 3.5, 2.5, 0.6$)

0.9522

ii. Find the value of σ such that there is a 2% chance of a wait time longer than 3.5 minutes.

Give your answer correct to two decimal places. 1 mark

$\Pr(Z \geq z) = 0.02$
 $\therefore \Pr(Z \leq z) = 0.98$
 $\therefore z = 2.0537\dots = \frac{3.5 - 2.5}{\sigma}$
 $\therefore \sigma = 0.49$

invNorm(0.98,0,1)

2.05375

$\text{solve}\left(2.053748910624 = \frac{3.5 - 2.5}{x}, x\right)$

$x = 0.486914$

d. The driver passes through three traffic lights (A , B and C) on their journey to work. The probability of each traffic light being red is shown in the table below.

Traffic light	A	B	C
Probability that the traffic light is red	0.2	0.3	0.1

Let Y be the random variable representing the number of traffic lights that are red on the driver’s journey to work. Assume that each traffic light being red is independent of any other traffic light being red.

Complete the following table for the probability distribution of Y . 2 marks

y	0	1	2	3
$\Pr(Y = y)$	$\frac{63}{125} = 0.504$	$\frac{199}{500} = 0.398$	$\frac{23}{250} = 0.092$	$\frac{3}{500} = 0.006$

GGG

GGR
 GRG
 RGR

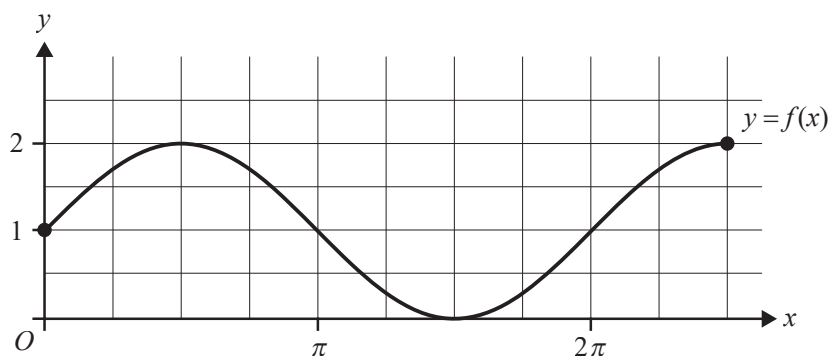
GRR
 RGR
 RRG

RRR

Question 4 (19 marks)

Consider the function $f: \left[0, \frac{5\pi}{2}\right] \rightarrow \mathbb{R}$, $f(x) = \sin(x) + 1$.

The graph of $y = f(x)$ is shown below.



- a. Evaluate $f\left(\frac{2\pi}{3}\right)$.

1 mark

$$f\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} + 1$$

$$f(x) = \sin(x) + 1 \mid 0 \leq x \leq \frac{5\pi}{2}$$

Done

$$f\left(\frac{2\pi}{3}\right)$$

$$\frac{\sqrt{3}}{2} + 1$$

- b. Find the exact values of x for which $f(x) = \frac{3}{2}$.

1 mark

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$$

$$\text{solve}\left(f(x) = \frac{3}{2}, x\right) \quad x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6} \text{ or } x = \frac{13\pi}{6}$$

- c. There exist real numbers a and k in the interval $\left(0, \frac{5\pi}{2}\right)$, such that $f(x+k) = f(x)$ for all $x \in [0, a]$.

Find the value of k and the largest possible value of a .

2 marks

$$\text{Period} = 2\pi$$

$$\therefore k = 2\pi$$

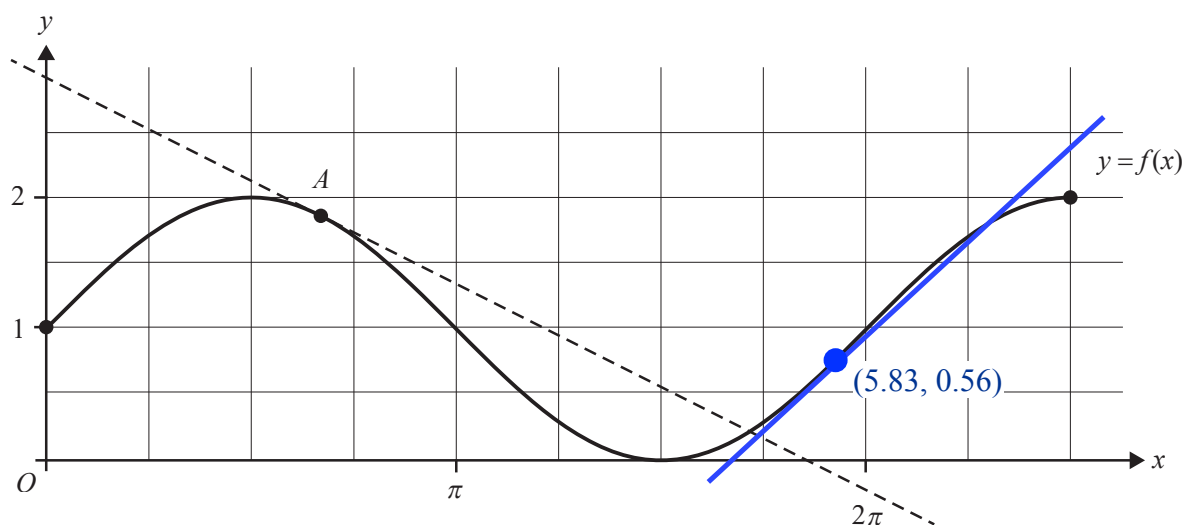
$$\text{no value of } x \text{ can go past } \frac{5\pi}{2}$$

$$\therefore a = \frac{5\pi}{2} - 2\pi$$

$$= \frac{\pi}{2}$$

Question 4 continues on the next page.

- d. Consider the tangent to the graph of $y = f(x)$ at the point A where $x = \frac{2\pi}{3}$, as shown on the axes below.



Find the equation of the tangent to the graph of $y = f(x)$ at the point where $x = \frac{2\pi}{3}$.

1 mark

$$y = -\frac{x}{2} + \frac{2\pi + 3(\sqrt{3} + 2)}{6}$$

$$\text{tangentLine}\left(f(x), \frac{2\pi}{3}\right) \\ \frac{2\pi + 3 \cdot (\sqrt{3} + 2)}{6} - \frac{x}{2}$$

- e. Apply two iterations of Newton's method to f with $x_0 = \frac{2\pi}{3}$.

- i. Write down x_2 , correct to **one decimal place**.

$$x_2 = 5.2$$

$$\frac{\frac{2\pi}{3} - \frac{f\left(\frac{2\pi}{3}\right)}{\frac{d}{dx}(f(x))\big|_{x=\frac{2\pi}{3}}}}{\frac{2\pi}{3} + \sqrt{3} + 2} \\ \frac{\frac{2\pi}{3} + \sqrt{3} + 2 - \frac{f\left(\frac{2\pi}{3} + \sqrt{3} + 2\right)}{\frac{d}{dx}(f(x))\big|_{x=\frac{2\pi}{3} + \sqrt{3} + 2}}}{5.20363}$$

1 mark

- ii. On the axes in **part d**, draw the tangent to the graph of $y = f(x)$ at the point where $x = x_1$.

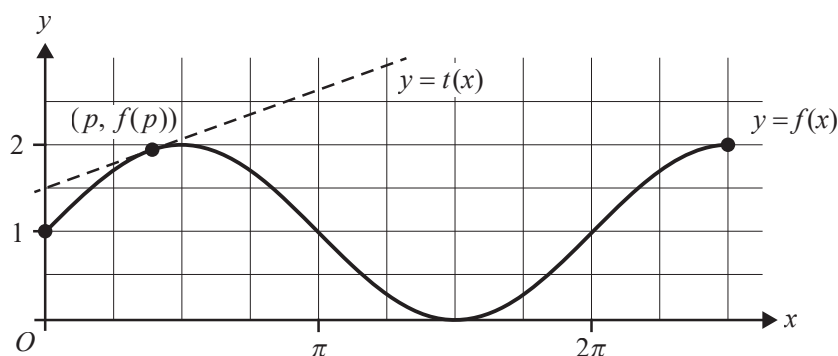
1 mark

(Answer on the graph in **part d**.)

Tangent at point with coordinates approximately (5.83, 0.56).

$$\frac{\frac{2\pi}{3} + \sqrt{3} + 2}{f\left(\frac{2\pi}{3} + \sqrt{3} + 2\right)} \\ 5.82645 \\ 0.558976$$

- f. Now consider the line $y = t(x)$, which is the tangent to the graph of $y = f(x)$ at the point $(p, f(p))$, where $p \in \left(0, \frac{5\pi}{2}\right)$.



- i. Show that $t(x) = \cos(p)(x - p) + \sin(p) + 1$.

2 marks

$$m_t = f'(p)$$

$$= \cos(p)$$

$$\text{at } (p, f(p))$$

$$t(x) = y$$

$$y - f(p) = \cos(p)(x - p)$$

$$y - (\sin(p) + 1) = \cos(p)(x - p)$$

$$y = \cos(p)(x - p) + (\sin(p) + 1)$$

$$\therefore t(x) = \cos(p)(x - p) + \sin(p) + 1$$

- ii. Determine the minimum and maximum possible values for the y-intercept of

$$y = t(x), \text{ for } p \in \left(0, \frac{5\pi}{2}\right).$$

2 marks

$$t(x) = x\cos(p) + (-p\cos(p) + \sin(p) + 1)$$

$$y\text{-int} = -p\cos(p) + \sin(p) + 1$$

$$\text{Values for } y\text{-int } \text{Max} = 1 + \pi \quad \text{Min} = 1 - 2\pi$$

$$\begin{aligned} & \text{fMax}(-p \cdot \cos(p) + \sin(p) + 1, p) | 0 < p < \frac{5 \cdot \pi}{2} \\ & p = \pi \\ & \text{fMin}(-p \cdot \cos(p) + \sin(p) + 1, p) | 0 < p < \frac{5 \cdot \pi}{2} \\ & p = 2 \cdot \pi \\ & -p \cdot \cos(p) + \sin(p) + 1 | p = \{\pi, 2 \cdot \pi\} \\ & \{\pi + 1, 1 - 2 \cdot \pi\} \end{aligned}$$

- iii. Determine the values of p for which $y = t(x)$ has a unique x-intercept that is equal to the x-intercept of $y = f(x)$.

Give your answers correct to two decimal places.

2 marks

$$p = 2.38, 7.04$$

(Note: Gradient of tangent $\neq 0$.)

Tangent is a horizontal line

i.e. no unique solution)

$$\begin{aligned} & \text{zeros}\left(t\left(\frac{3 \cdot \pi}{2}\right), p\right) | 0 < p < \frac{5 \cdot \pi}{2} \text{ and } \cos(p) \neq 1 \\ & \{2.381, 7.044\} \end{aligned}$$

Question 4 continues on the next page.

g. Let $g: \left[0, \frac{5\pi}{2}\right] \rightarrow \mathbb{R}$, $g(x) = ax^3 + bx^2 + cx + d$ be a polynomial function,

where $a, b, c, d \in \mathbb{R}$.

Suppose $g(0) = f(0)$ and $g'(0) = f'(0)$.

i. Show that $c = 1$ and $d = 1$.

2 marks

$g(0) = d$	_____	$g'(x) = 3ax^2 + 2bx + c$	_____
$f(0) = \sin(0) = 1$	_____	$g'(0) = c$	_____
$= 1$	_____	$f'(0) = \cos(0)$	_____
Since $f(0) = g(0)$	_____	$= 1$	_____
$d = 1$	_____	Since $f'(0) = g'(0)$	_____
	_____	$c = 1$	_____

ii. If $g(2\pi) = f(2\pi)$ and $g'(2\pi) = f'(2\pi)$, determine the area bounded by the graphs of $y = f(x)$ and $y = g(x)$, for $x \in [0, 2\pi]$.

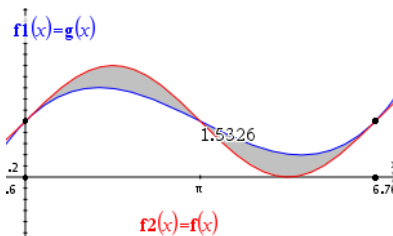
Give your answer correct to two decimal places.

2 marks

$$a = \frac{1}{2\pi^2} \quad b = \frac{-3}{2\pi}$$

$$\text{Area bound} = \int_0^{\pi} (f(x) - g(x)) dx + \int_{\pi}^{2\pi} (g(x) - f(x)) dx$$

$$= 1.53 \text{ square units.}$$



$$g(x) := a \cdot x^3 + b \cdot x^2 + x + 1$$

Done

$$\text{solve} \left(\begin{cases} f(x) = g(x) \\ \frac{d}{dx}(f(x)) = \frac{d}{dx}(g(x)) \end{cases}, \{a, b\} \right) | x = 2 \cdot \pi$$

$$a = \frac{1}{2 \cdot \pi^2} \text{ and } b = \frac{-3}{2 \cdot \pi}$$

$$a := \frac{1}{2 \cdot \pi^2} \quad \frac{1}{2 \cdot \pi^2}$$

$$b := \frac{-3}{2 \cdot \pi} \quad \frac{-3}{2 \cdot \pi}$$

$$\int_0^{\pi} (f(x) - g(x)) dx + \int_{\pi}^{2\pi} (g(x) - f(x)) dx = 1.5326$$

iii. Let $a = 0$, $c = 1$, $d = 1$.

Find b and r , such that $g(r) = f(r)$ and $g'(r) = f'(r)$, where $b \in \mathbb{R}$

and $r \in \left(0, \frac{5\pi}{2}\right)$.

2 marks

Calculator doesn't give exact solution, but obviously $r = \pi$.

Solve $f(\pi) = g(\pi)$ for b .

If $r = \pi$, $b = \frac{-1}{\pi}$.

$$f(x) := \sin(x) + 1 | 0 \leq x \leq \frac{5 \cdot \pi}{2}$$

Done

$$g(x) := b \cdot x^2 + x + 1$$

Done

$$\text{solve} \left(\begin{cases} f(x) = g(x) \\ \frac{d}{dx}(f(x)) = \frac{d}{dx}(g(x)) \end{cases}, \{x, b\} \right) | 0 < x < \frac{5 \cdot \pi}{2}$$

7 and $b = 0$. or $x = 3.14159$ and $b = -0.31831$

$$\text{solve}(f(\pi) = g(\pi), b)$$

$$b = \frac{-1}{\pi}$$

End of examination questions