VCE General Mathematics Teacher Resource Book *for*

TI-Nspire™ CX II CAS graphing calculator

Teachers Teaching with Technology™
Professional Development from Texas Instruments

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Texas Instruments

T³ Teachers Teaching with Technology™

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The Teachers Teaching with Technology™ (T³™) Australia professional learning organization is comprised of some of the most creative and innovative mathematics and STEM teachers in the world. They are dynamic and passionate educators who share their knowledge and expertise with secondary teachers and students through professional development events and resource creation.

Contents

Introduction

This publication, *VCE General Mathematics Teacher Resource Book for the TI-Nspire™ CX II CAS*, is intended to support senior secondary school mathematics teachers in Victoria as they seek to teach the *VCAA Mathematics Study Design 2023-2027*.

Specifically, the publication highlights ways in which *TI-Nspire CAS* technology might be used to assist in the teaching, learning and assessment of *VCE General Mathematics Units 1 to 4*.

It is not a complete manual for using this technology, rather it tries to look at each syllabus dot point and make suggestions for possible classroom use.

It has been developed by experienced educators and reviewed by senior mathematics teachers from Victorian schools. We hope you find this to be a useful and supportive publication.

*[**Note:** A digital version of this publication can be found at https://education.ti.com/aus/VIC].*

Notes for teachers

To maximise the usefulness of *VCE General Mathematics Teacher Resource Book for the TI-Nspire™ CX II CAS* to teachers, the authors have provided the following explanatory notes.

- It is assumed that the user of this teacher resource book has a basic familiarity with navigating calculator documents and pages. Readers requiring an introduction to this are referred to tutorials at https://education.ti.com/aus and https://www.youtube.com/@TIAustralia.
- Throughout this publication, unless otherwise specified, the default calculator document settings have been used. The calculator user interface language has been set to *English (U.K.)*.
- For each example task, it is desirable to start a new calculator document ([ctrl] [N] or [on] [1]). Alternatively, insert a new problem ([doc▾] > **Insert** > **Problem**).
- When working with functions, use of the **assign** command (via '**:=**') has been privileged over the **define** command. While both commands essentially perform the same role, the **assign** command is a more natural command to use in the **Notes** application.
- Implied multiplication has been assumed when working with products such as $6x$. However, where it is necessary to use the multiplication key when entering the product *bx*, for example, the symbol '$*$' or '$\times$' is used to denote multiplication.
- In some instances in this publication, space has been added to the syntax of commands to improve readability, even though in general spaces should **not** be used in calculator commands without a clear reason. For example, when entering a function, the authors may have expressed this as ***f*(*x*) := *a* + *bx*,** but on the calculator, it will appear as *f*(*x*):=*a*+*b*·*x*.
- There will be some variation in the formatting of commands and text to be entered, but the authors have attempted to use bold formatting when referring to commands to be entered or accessed via the calculator.
- For screenshots from the **Graphs** Application, grid and label settings will vary. Use [menu] commands (or [ctrl] [menu]) to modify these settings for an open document. The default settings (for all documents) for grids and labels can be edited by pressing [menu] and then select **Settings**.
- When catalogue commands are mentioned, pressing [📖] and then [1] will display catalog commands in alphabetical order. Pressing the first letter of the desired command will locate it more quickly.
- To make this publication as practical and concise as possible, mathematical problems considered have been restricted to those that can be attempted by teachers and students without using pre-prepared files. For more interactive digital resources aligned to *VCE General Mathematics Units 1 to 4*, go to https://education.ti.com/aus.

VCE General Mathematics Unit 1

1.1. Investigating and comparing data distributions

1.1.1. Displaying the distribution of a numerical variable

Plotting the distribution of a numerical variable

Eighteen members of a youth group are lining up for tickets to an 'Australian Idol' concert. Their current ages are: 10, 13, 10, 15, 16, 11, 11, 16, 15, 10, 10, 11, 19, 16, 15, 11, 10, 15.

(a) Represent the current ages of the members in a dot plot.

(b) Comment on the variation within the data, and what other information the dot plot reveals.

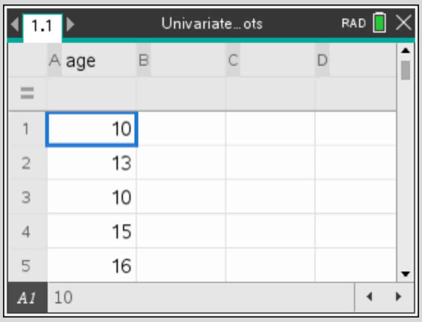
(c) Represent the current ages as a box plot.

(d) Represent the current ages as a histogram with an interval width of two years.

(a) On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable name ***age***.
- Enter the data for ***age*** into column A.

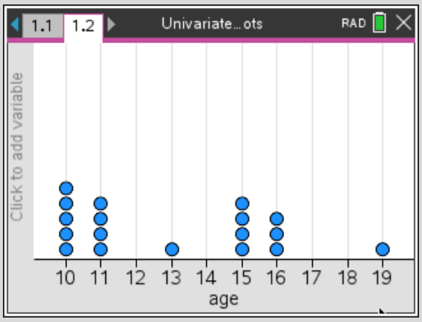
__Note:__ There are shortcuts for moving the cursor more quickly around a __Lists & Spreadsheet__ page.
Press [ctrl] [1] to go to the last entry in a column.
Press [ctrl] [7] to go to the first entry in a column.
Press [ctrl] [3] to go down a page and [ctrl] [9] to go up a page.



Add a **Data & Statistics** page, and then:

- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select the variable ***age***.

The default plot is a dot plot.



(b) Answer: The dot plot shows that:

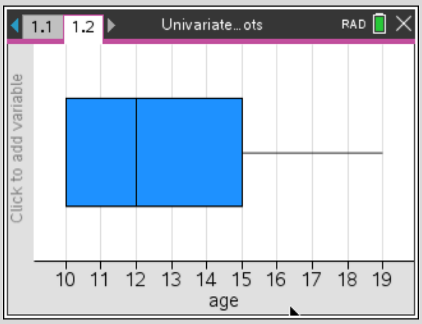
- Student ages range from 10 to 19.
- The most common age of these students is 10, but a few other ages (11, 15) occur nearly as frequently.
- Half of the students are aged 10 or 11 years old.
- One student is at least three years older than any other student.

(c) To change the plot type to a box plot:

- Press [menu] > **Plot Type** > **Box Plot**.

Moving the cursor over the box plot will confirm the quartile values for the student ages.

__Note:__ The plot type can also be changed by hovering over the plot window, and then pressing [ctrl] [menu].
__Note:__ The default box plot will display outliers if any exist. To hide the display of outliers, hover over the box plot and then press [ctrl] [menu], then select __Extend Box Plot Whiskers__.



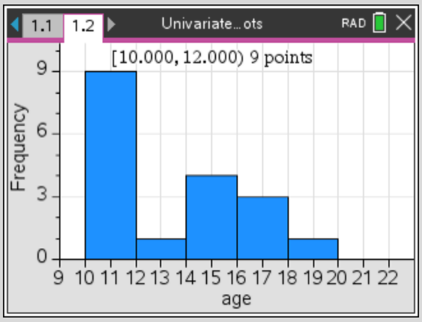
... continued

Plotting the distribution of a numerical variable (continued)

(d) To change the plot type to a histogram:

- Press [menu] > **Plot Type** > **Histogram**.
- Press [menu] > **Plot Properties** > **Histogram Properties** > **Bin Settings** > **Equal Bin Width**.
- Change **Width** to 2.
  (i.e. the width of each histogram column)
- Change **Alignment** to 10.
  (i.e. the starting age value for the histogram columns)
- Press [menu] > **Window/Zoom** > **Zoom – Data**.

Moving the cursor over the histogram will confirm the frequencies for each age.



Plotting a histogram when using variables with frequencies

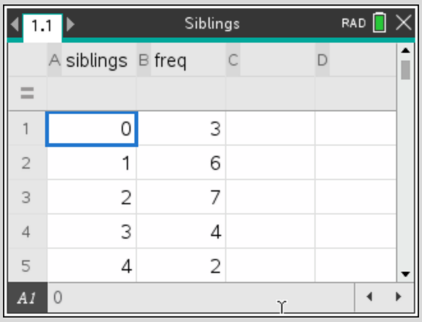
The number of siblings for each of the 26 students in a Year 11 class were recorded in a table.

No. of siblings	0	1	2	3	4	5	6
Frequency	3	6	7	4	2	3	1

Display this table as a histogram.

On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable name ***siblings***.
- In the column B heading cell, enter the name ***freq***.
- Enter the data for ***siblings*** into column A.
- Enter the frequency information into column B.



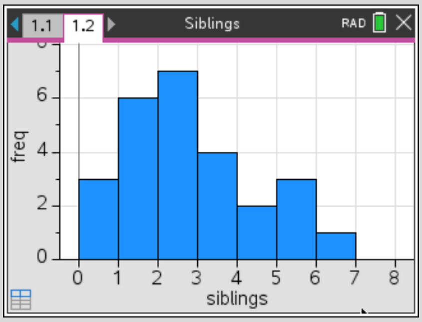
Add a **Data & Statistics** page, and then:

- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select the variable ***siblings***.

The default plot is a dot plot.

- Press [menu] > **Plot Properties** > **Add Y Summary List**.
- Select ***freq*** from the pop-up menu.

Note: *For scenarios involving continuous numerical variables that have intervals, use the midpoint of the intervals as the 'typical' value.*



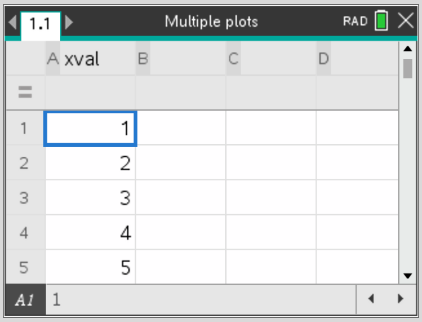
Comparing plot types of the distribution of a numerical variable

Compare the use of a dot plot and a box plot for displaying the distribution of the variable *xval*, if *xval* is the following set of twenty numbers: 1, 2, 3, 4, 5, 6, 6, 6, 6, 7, 7, 8, 8, 8, 8, 9, 9, 10, 10, 10.

On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable name ***xval***.
- Enter the data for ***xval*** into column A.

Note*: The variable name 'xval' is used here to avoid any confusion with the spreadsheet column labelled 'X'.*

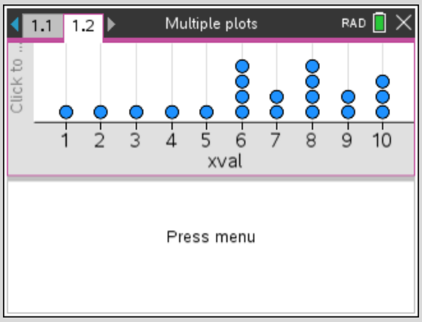


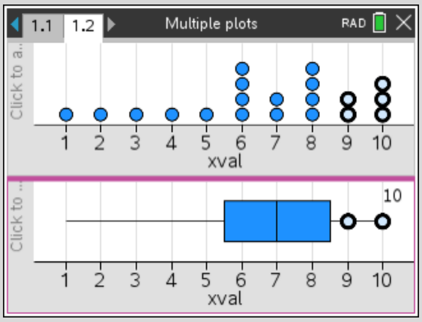
Add a **Data & Statistics** page, and then:

- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select the variable ***xval***.

The default plot is a dot plot. To split the current page and add a box plot of *xval* below the dot plot:

- Press [doc▾] > **Page Layout** > **Select Layout** > **Layout 3**.
- Click in the lower half of the screen and press [menu], then select **Add Data & Statistics**.
- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select the variable ***xval***.
- Press [menu] > **Plot Type** > **Box Plot**.





Hover over the data. If you click on any points in one plot, the corresponding points are highlight in both plots. This makes clearer how each plot type represents the distribution of *xval*. For example, note how the top 25% of values is displayed differently in the dot plot and box plot (see right).

Note: *It is possible to 'move' a point (or points), and examine how each plot is affected. To do this, select the required point(s), then drag the point(s) to a different location. Remember that* [ctrl] [esc] *will undo such changes. Click away from the selected point(s) to deselect them.*

Constructing parallel box plots

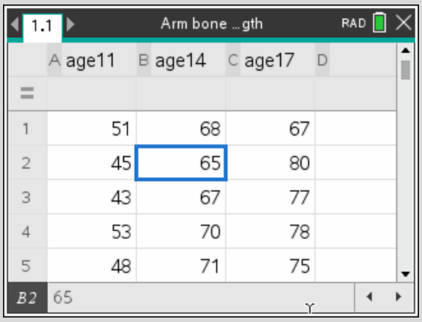
Three adolescents failed to return from a fishing trip. They were Maria aged 11, Thanh aged 14, and Sharma aged 17. Later, a shark was caught offshore, which had a human arm in its stomach. The arm had no identifying features, but the length could be measured. Police found it to be 66 cm.
As part of the investigation, data was collected about the arm lengths of students of the three ages.

Age 11	51	45	43	53	48	48	59	50	50	47	51	49	48	51	47	50
Age 14	68	65	67	70	71	67	66	71	59	62	65	62	67	64	62	60
Age 17	67	80	77	78	75	78	74	76	77	71	65	66	73	66	80	74

Use parallel box plots to help decide whether the arm belongs to one of the missing people.

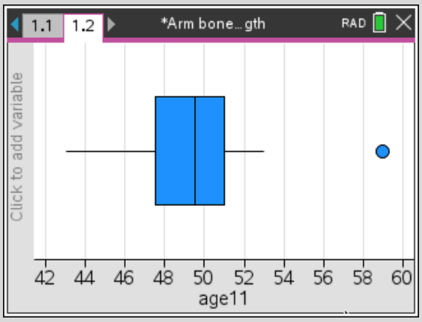
On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the name ***age11***.
- In the column B heading cell, enter the name ***age14***.
- In the column C heading cell, enter the name ***age17***.
- Enter the data for the three variables into columns A, B and C respectively.



Add a **Data & Statistics** page, and then:

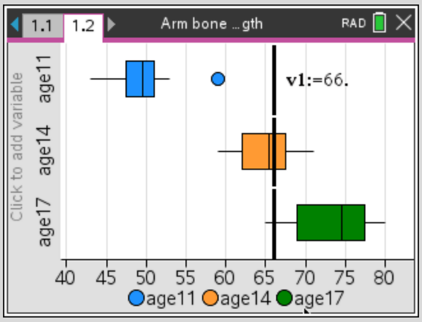
- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select the variable ***age11***.
- Press [menu] > **Plot Type** > **Box Plot**.
- Press [menu] > **Plot Properties** > **Add X Variable** and select the variable ***age14***.
- Repeat the previous step to add a box plot for the variable ***age17***.



Note: *The plot type can also be changed by hovering over the plot window, and then pressing* [ctrl] [menu].

To plot the arm bone length of 66 cm:

- Press [menu] > **Analyse** > **Plot Value**.
- Enter the value **66**.



Answer: Observing the parallel box plots and the plotted value of 66 cm, it is possible, but unlikely, that the arm belongs to the 11-year-old, as the maximum arm length for 11-year-olds is 59 cm. A 66 cm arm is in the lowest quartile of the 17-year-old data and so quite possible that the arm belongs to the 17-year-old. It is most likely that it is a 14-year-old's arm, as the median arm length for this group is 65.5 cm. However, more data (including gender specific data) is needed to make a more confident prediction.

Note: *The plotted value can be removed by pressing* [menu] > ***Analyse > Remove Plotted Value****. A box plot can be removed by pressing* [menu] ***> Plot Properties > Remove X Variable****.*

1.1.2. Summarising the distribution of a numerical variable

Calculating individual statistics for a numerical variable

The handspans (to the nearest cm) of a sample of ten students were recorded as follows:

15, 14, 17, 19, 14, 14, 13, 16, 21, 16.

Calculate the mean, median and standard deviation of the handspans of students in the sample.

On a **Calculator** page:

- Type the variable name ***handspan***.
- Press [ctrl] [:=] to enter the 'Assign' symbol.
- Press [ctrl] [)] to enter the braces (set brackets).
- Enter the ***handspan*** data as shown.

To calculate the mean of the sample:

- Press [menu] **> Statistics > List Maths > Mean**.
- Press [var] and select the variable ***handspan***.

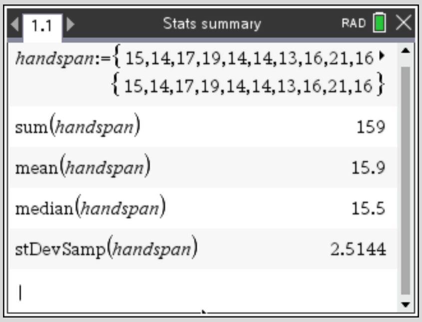
To calculate the median of the sample:

- Press [menu] **> Statistics > List Maths > Median**.
- Press [var] and select the variable ***handspan***.

To calculate the standard deviation of the sample:

- Press [menu] **> Statistics > List Maths > Sample Standard Deviation**.
- Press [var] and select the variable ***handspan***.

Answer: See statistical results in screen shown right.



***Note**: If the calculator is set to **Auto Calculation Mode**, press* [ctrl] [enter] *to express answers in decimal format.*
*If the calculator is set to **Approximate Calculation Mode**, the answers will default to decimal format.*
*The **Calculation Mode** can be set via* [on] ***> Settings > Document Settings**.*

Summarising a numerical variable from a Calculator page

The handspans (to the nearest cm) of a sample of ten students were recorded as follows:

15, 14, 17, 19, 14, 14, 13, 16, 21, 16.

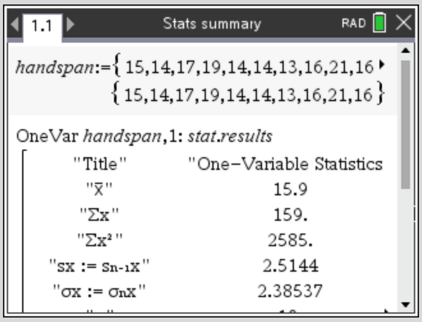
Calculate the summary statistics for the handspans of students in the sample.

On a **Calculator** page:

- Type the variable name ***handspan***.
- Press [ctrl] [:=] to enter the 'Assign' symbol.
- Press [ctrl] [)] to enter the braces (set brackets).
- Enter the ***handspan*** data as shown.

To calculate the summary statistics of the sample:

- Press [menu] **> Statistics > Stat Calculations > One Variable Statistics**.
- Set the following values:
  - **Num of Lists = 1**.
  - **X1 List,** click ***handspan***.
- Press [enter] to calculate and display the summary statistics for ***handspan***. Use the arrow keys to scroll through the summary statistics.



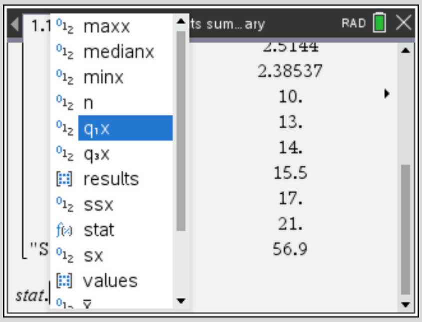
continued …

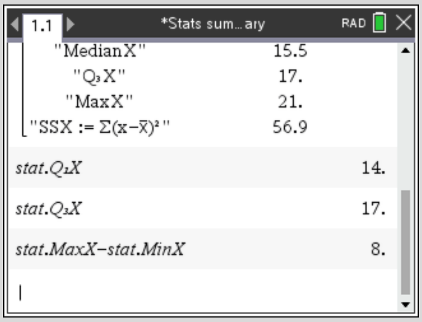
Summarising a numerical variable from a Calculator page (continued)

Once the summary statistics have been calculated, they can be accessed individually using the following procedure:

- Type **stat.** (including the decimal point symbol)
- Select the required summary statistic from the pop-up list (e.g. the value of Q1).
- Press [enter] twice to display the value of the summary statistic.

***Note:** All summary statistics are stored and accessible but are only relevant to the most recent calculation of summary statistics. They are updated after any recalculation of the summary statistics*





Summarising a numerical variable from a Lists & Spreadsheet page

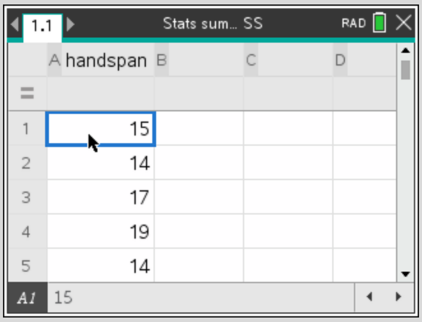
The handspans (to the nearest cm) of a sample of ten students were recorded as follows:

15, 14, 17, 19, 14, 14, 13, 16, 21, 16.

Calculate the summary statistics for the handspans of students in the sample.

On a **Lists & Spreadsheet** page:

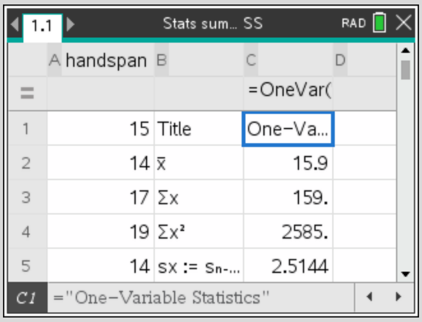
- In the column A heading cell, enter the variable name ***handspan***.
- Enter the data for ***handspan*** into column A.



- Press [menu] > **Statistics** > **Stat Calculations** > **One Variable Statistics**.
- Set the following values:
  - **Num of Lists = 1**.
  - **X1 List** = ***handspan***.
  - **1st Result Column** = **b[]** (to place statistical results starting from column B).
- Press [enter] to calculate and display the summary statistics for ***handspan***.

Use the arrow keys to scroll through the summary statistics.

***Note**: The summary statistics that have been calculated most recently can be accessed from a **Calculator** page (as above).*



Summarising a numerical variable with frequency information

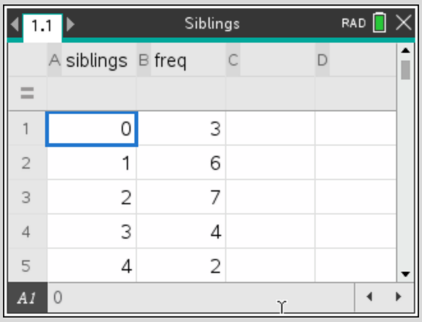
The number of siblings for each of the 26 students in a Year 11 class were recorded in a table.

No. of siblings	0	1	2	3	4	5	6
Frequency	3	6	7	4	2	3	1

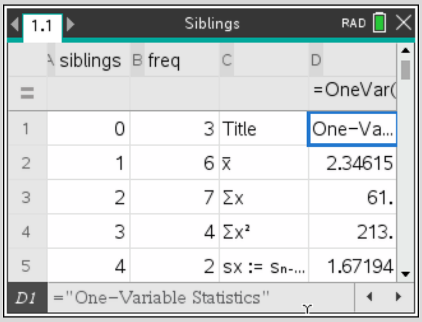
Find the mean number of siblings for students in the class, correct to two decimal places.

On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable name ***siblings***.
- In the column B heading cell, enter the name ***freq***.
- Enter the data for ***siblings*** into column A.
- Enter the frequency information into column B.



- Press menu > **Statistics** > **Stat Calculations** > **One Variable Statistics.**
- Set the following values:
  - **Num of Lists** = **1**.
  - **X1 List** = ***siblings***.
  - **Frequency List** = ***freq***.
  - **1st Result Column** = **c[]** (to place statistical results starting from column C).
- Press enter to calculate and display the summary statistics for ***siblings***.



Use the arrow keys to scroll through the summary statistics.

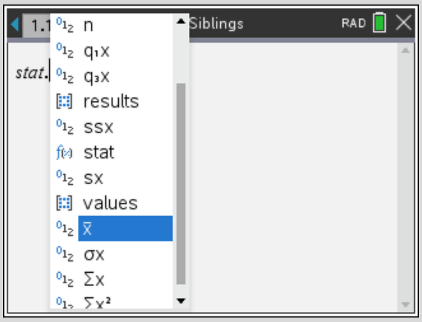
Answer: The mean number of siblings for students in the class is 2.35, correct to two decimal places.

The summary statistics that have most recently been calculated can be accessed individually.

To do this, add a **Calculator** page and then:

- Type **stat.** (including the decimal point symbol) to display a pop-up menu of the calculated statistics.
- Select the desired statistic and press enter twice.

To display all the calculated statistics, enter **stat.results**.



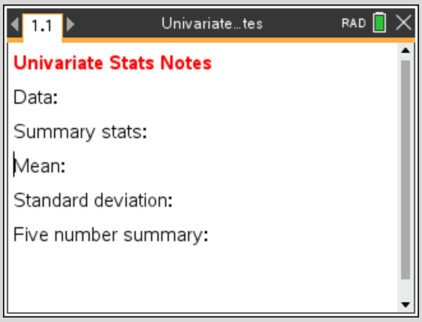
Note: *For scenarios involving continuous numerical variables that have intervals, use the midpoint of the intervals as the 'typical' value. In such scenarios the summary statistics calculated will be estimates of the actual statistics.*

Creating a Notes page to summarise a numerical variable

A **Notes** page can be constructed to automatically calculate summary statistics for a numerical variable. We will use sample data as follows: 10, 13, 10, 15, 16, 11, 11, 16, 15.

On a **Notes** page:

- Enter the text shown in the screenshot (the colon symbol can be found via the [?!▸] key).
- Move the cursor to the right of the word 'Data' and press [menu] > **Insert** > **Maths Box** (or press [ctrl] [M]).



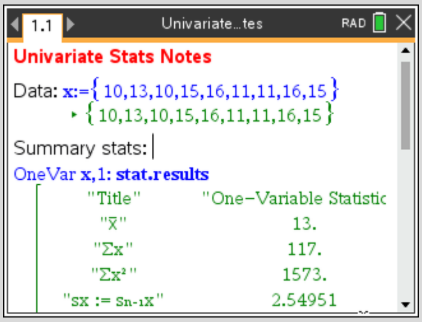
Repeat to insert **Maths Boxes** next to each of the other template headings.

Note: *To edit the text colour, select the text by holding* [⇧shift] *and cursor keys. Then press* [menu] *>* ***Format*** *>* ***Text colour****.*

- Click on the **Maths Box** next to the word 'Data'.
- Inside the **Maths Box**, input using assign ([ctrl] [▫:=▫]) and braces ([ctrl] [ )]) as follows:

  $x := \{10, 13, 10, 15, 16, 11, 11, 16, 15\}$

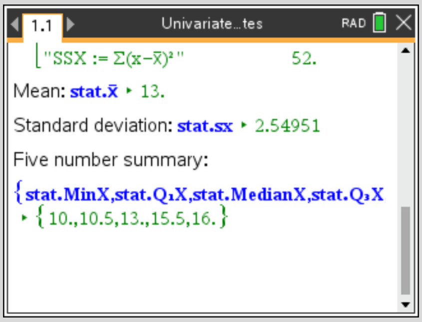
- For 'Summary stats:', press [menu] > **Calculations** > **Statistics** > **Stat Calculations** > **One–Variable Statistics …** and then set the following values:
  - **Num of Lists = 1.**
  - **X1 List** = x
- Press [enter] twice to calculate and display the summary statistics for x.



In the other **Maths Boxes**:

- For 'Mean', type **stat.** and then select $\bar{x}$.
- For 'Standard deviation', type **stat.** and then select sx.
- For 'Five number summary', press [ctrl] [ )] to enter the braces (set brackets), and then use the above method to complete the following expression:

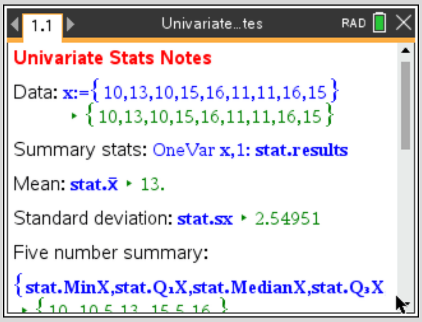
$$\{stat.MixX, stat.Q1X, stat.MedianX, stat.Q3X, stat.MaxX\}$$



Note*: The statistical variables from the most recent statistical calculations can also be accessed via the* [var] *key.*

To hide the output results from the 'Summary stats' maths box, click inside that maths box and then:

- Press [menu] > **Maths Box Options** > **Maths Box Attributes**.
- In the **Input & Output** attribute, select **Hide Output**.



This allows the most relevant stats to be displayed more conveniently. If the data values for x are changed, the statistics will be updated automatically.

Note*: Changes to the attributes of a* ***Maths Box*** *are not always visible until you have clicked outside of that* ***Maths Box****. For example, the* ***Hide Output*** *attribute selected above is not enacted until the user clicks outside the 'Summary stats'* ***Maths Box****.*

Investigating the meaning of standard deviation

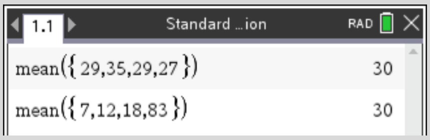
Jemima and Hania play for a local cricket club. Their first four lots of scores for the season are:

- Jemima: 29, 35, 29 and 27
- Hania: 7, 12, 18, and 83

For each batter, how does each score deviate from their mean score? Is there a way to compare the scores for each cricketer and look at the ways their scores are similar and the ways in which they are different? It also explores the usefulness of the 'standard deviation'.

To calculate the mean scores for Jemima and Hania, on a **Calculator** page:

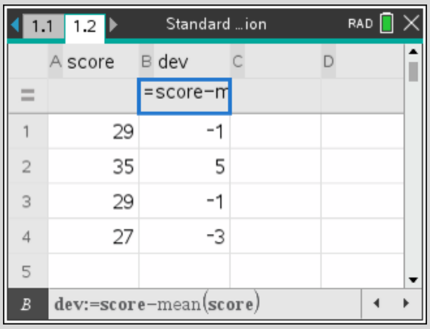
- For Jemima's mean score, enter **mean({29,35,29,27})**.
- For Hania's mean score, enter **mean({7,12,18,83})**.



The mean scores are the same (both mean scores are 30 runs).

To calculate by how much each score deviates from the mean score for Jemima, on a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the label ***score***.
- In the column B heading cell, enter the variable name ***dev***.
- Enter the data for ***score*** into column A.
- In the column B formula cell, enter the formula ***=score – mean(score)***.



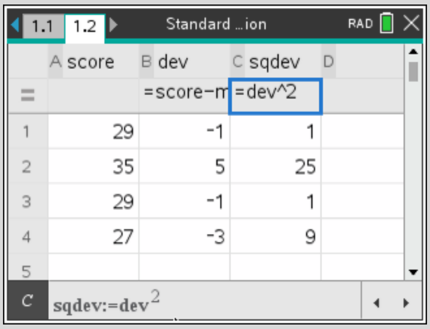
The sum of these 'deviations' is zero ($-1+5+-1+-3=0$), which is unhelpful as the 'average deviation' of the scores (This will always be the case).

The 'negative deviations' can be made positive by squaring the values of the deviations. To calculate the 'squared deviations' for Jemima, on a **Lists & Spreadsheet** page:

- In the column C heading cell, enter the label ***sqdev***.
- In the column C formula cell, enter the formula ***=dev²***.

The sum of the 'squared deviations' is 36.

$$\left(\text{i.e. } (-1)^2+(5)^2+(-1)^2+(-3)^2=36\right)$$

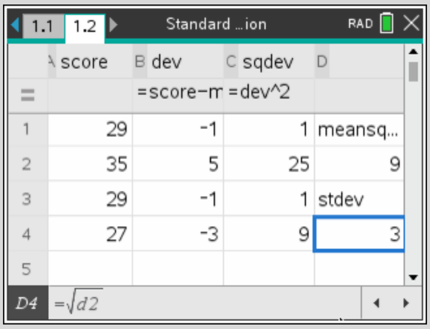


The average or mean 'squared deviation' can be calculated as follows:

- Click in the cell D1, and enter the label **meansqdev**.
- Click in the cell D2, and enter the formula **=mean(sqdev)**.

Finally, to find the average deviation (called the 'standard deviation') by this method, find the square root of the value of the average squared deviation as follows:

- Click in the cell D3, and enter the label **stdev**.
- Click in the cell D4, and enter the formula **=sqrt(D2)**.



Now we have a *standardised* measure of the mean deviation (referred to as the standard deviation), which ignores whether the individual deviations are positive or negative.

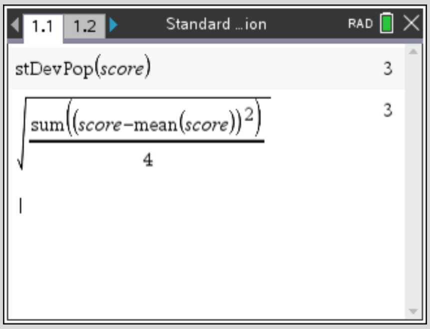
... continued

Investigating the meaning of standard deviation (continued)

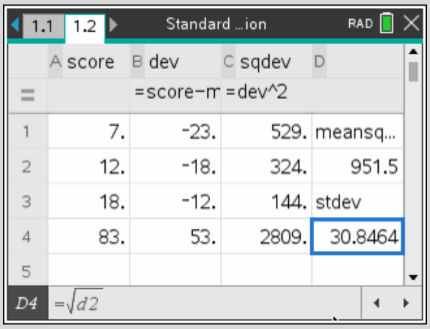
To verify this measure, on a **Calculator** page:

- Press menu **> Statistics > List Maths > Population Standard Deviation**, then enter **stDevPop(*score*)** as shown.
- Enter the formula summary

$$\sqrt{\frac{\textbf{sum}\left(\left(score - \textbf{mean}(score)\right)^2\right)}{\textbf{4}}}$$



For comparison, if Hania's scores are entered (see screen at right), the standard deviation is much greater (approximately 31 runs), reflecting greater variability in her scores.



Further exploration using the spreadsheet might include finding four scores that have a particular standard deviation, or the 'smallest' and 'largest' possible standard deviation.

Note: *The general formula for the standard deviation of a numerical variable x which has n values is:*

$$\textbf{\textit{Standard deviation}} = \sqrt{\frac{\sum\left(x-\bar{x}\right)^2}{n}}$$

*In senior mathematics courses where the standard deviation of a sample is used as an estimate of the population standard deviation, it is usual to use **n – 1** rather than **n** in the definition.*

1.2. Arithmetic and geometric sequences, first-order linear recurrence relations and financial mathematics

Note: *This section demonstrates a variety of techniques for generating and displaying sequences using TI-Nspire CX II CAS. These techniques are applicable to any first-order recurrence relation or its explicit rule, irrespective of whether the sequence is arithmetic or geometric.*

1.2.1. Generating a sequence from a recurrence relation or explicit rule

Generating the terms of a sequence recursively in the Calculator application

A company purchases an asset for $34,000. For taxation purposes, there are two permissible options for calculating the declining value of the asset after n years. The recurrence relations that model these options are $u_0 = 34000,\ u_{n+1} = u_n - 3600$ and $v_0 = 34000,\ v_{n+1} = 0.85 \times v_n$.

(a) Use a recursive process in the Calculator application to generate a sequence with rule:

(i) $u_0 = 34000,\ u_{n+1} = u_n - 3600,\ n = \{0,1,\ldots,10\}$ (option 1).

(ii) $v_0 = 34000,\ v_{n+1} = 0.85 \times v_n,\ \ n = \{0,1,\ldots,10\}$ (option 2).

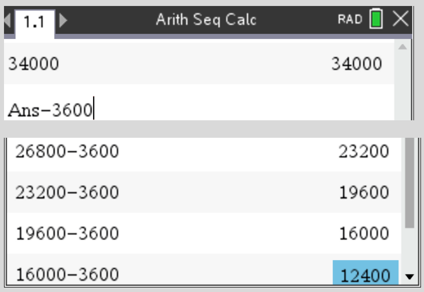
(b) Compare the value of the asset after 6 years for the two options (to the nearest dollar).

(c) For the recurrence relation $v_0 = 34000,\ v_{n+1} = 0.85 \times v_n,\ n = \{0,1,\ldots,10\}$, generate both the values of the sequence and the corresponding time periods. Confirm the value of term v_6 (to the nearest dollar).

(a)(i) To generate the values of the sequence with recurrence relation $u_0 = 34000,\ u_{n+1} = u_n - 3600$, on a **Calculator** page:

- Enter **34000**
- Press [–][3][6][0][0] (appears as **Ans–3600**), then [enter].
- Press [enter] to generate each new value of the sequence.

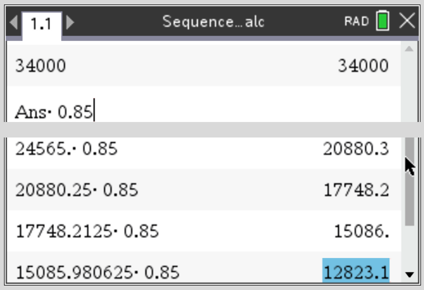
Answer: After 6 years, the value is $12,400 $\left(u_6 = 12\,400\right)$.



(a)(ii) To generate the values of the sequence with recurrence relation $v_0 = 34000,\ v_{n+1} = 0.85 \times v_n$, on a **Calculator** page:

- Enter **34000**
- Press [×][0][.][8][5] (appears as **Ans·0.85**), then [enter].
- Press [enter] to generate each new value of the sequence.

(b) Answer: After 6 years, the value using option 2 is $12,823, compared with $12,400 for option 1.

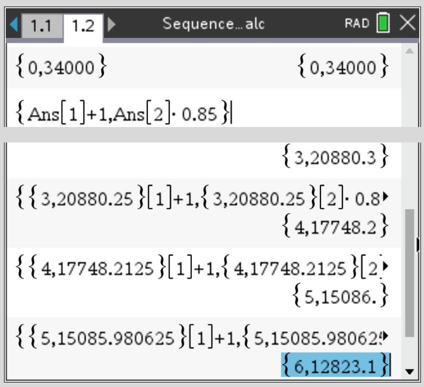


(c) To generate both the value of the sequence, v_n, and the corresponding time periods, n, on a **Calculator** page:

- Enter **{0,34000}**, being a list of two elements {time,value}
- Enter **{Ans[1]+1,Ans[2]×0.85}**, pressing [ctrl] [(–)] ([ans]) to key in **Ans.** This input adds 1 to the first element of previous list and multiplies the second element by 0.85.
- Press [enter] to generate each new term of the sequence.

Answer: The output $\{n, v_n\} = \{6, 12823.1\}$ confirms that the value after 6 years is $12,823.

Note: *For option 1, enter* ***{Ans[1]+1,Ans[2]–3600}*** *instead.*



Generating a sequence as a list in the Calculator application using the 'seq' command

The recurrence relations $u_0 = 34000$, $u_{n+1} = u_n - 3600$ and $v_0 = 34000$, $v_{n+1} = 0.85 \times v_n$ have explicit rules $u_n = 34000 - 3600n$ and $v_n = 34000 \times (0.85)^n, n = \{0,1,...\}$. These rules model two different options for calculating the value of an asset, n years after it is purchased for $34,000.

(a) In the Calculator application, use the 'seq' command to generate a list of terms for the sequence with rule:

(i) $u_n = 34000 - 3600n,\ n = \{0,1,...,10\}$ **(ii)** $v_n = 34000 \times (0.85)^n, n = \{0,1,...,10\}$

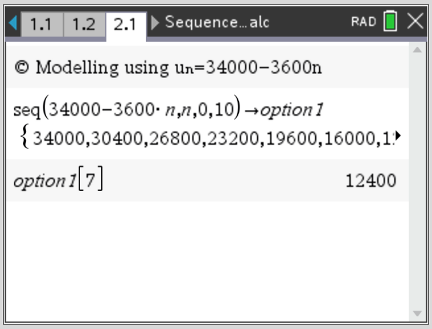
(b) Compare the values of the elements of these lists with the values from the previous problem.

(a) To generate a list of terms for the sequence with rule $u_n = 34000 - 3600$, on a **Calculator** page:

- Press [book] [S] and navigate to **seq**. Note the syntax is **seq**(Expr, Var, Low, High [,Step]). The default Step = 1.
- Key in $\mathbf{seq(34000 \times 0.85^n, n, 0, 10)}$. Do not press [enter].

To store the sequence with list name **option1**:

- Move the cursor to the right of the last bracket, then press [ctrl] [var] ([sto→]) and enter the list name **option1**, as shown.
- Press [var] to select '**option1**' and enter **option1[7]**.



***Note**: The elements in the option 1 list are numbered as follows:*

option1[1] = u_0 =34 000
option1[2] = u_1 =30 400
…
option1[7] = u_6 =12 400.

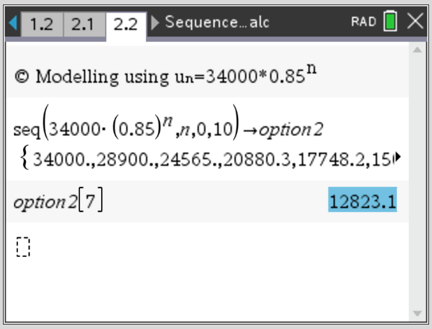
(b)(i) Answer: Both sequences show $12,400 after 6 years.

(a)(ii) To generate a list of terms for the sequence with rule $u_n = 34000 \times 0.85^n$, on a **Calculator** page:

- Press [book] [1] [S] and navigate to **seq**.
- Key in $\mathbf{seq(34000 \times 0.85^n, n, 0, 10)}$. Do not press [enter].

To store the sequence with list name **option2**:

- Move the cursor to the right of the last bracket, then press [ctrl] [var] ([sto→]) and enter the list name **option2**, as shown.
- Press [var] to select '**option2**' and enter **option2[7]**.



(b)(ii) Answer: Both sequences show $12,823.10 after 6 years.

Generating a sequence recursively in the Lists & Spreadsheet application

Ursa and Victor are training for a cycle race over 13 days. Ursa initially cycles 20 km. Each subsequent day, Ursa cycles 3 km more than the previous day. Victor initially cycles 16 km. Each subsequent day, Victor cycles 10% further than the previous day.

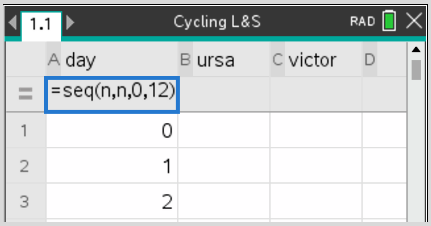
In the **Lists & Spreadsheet** application:

(a) Use the 'seq' command to generate a sequence of training days, $n = \{0, 1, ..., 12\}$.

By recursively filling down appropriate formulas to generate a sequence of training distances cycled by **(i)** Ursa, **(ii)** Victor, determine the distance for each cyclist on the 13th day.

(a) To set up the sequences, on a **Lists & Spreadsheet** page:

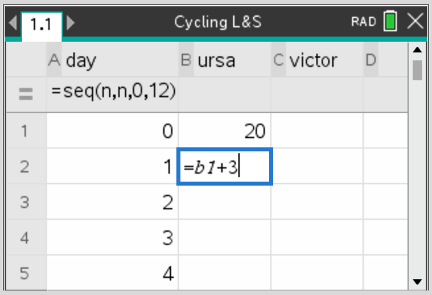
- In the heading cells, enter **day, ursa** and **victor**, as shown.
- In the column **A** formula cell (second from the top), enter the formula, **=seq(*n*,*n*,0,12)**, by pressing [📖] [S] and navigating to **seq**. Note the syntax for the command is: **seq(Expr, Var, Low, High [,Step]).**
  The default value of **Step = 1**.



(b)(i) To generate the sequence, $u_0 = 20,\ u_{n+1} = u_n + 3$:

- In cell **B1**, enter **20**, as shown.
- In cell **B2**, enter the formula, **=*b*1+3**, and then press [enter].
- Return to cell **B2** and press [ctrl] [menu] > **Fill**.
- Press ▼ to cell **B13** then [enter].

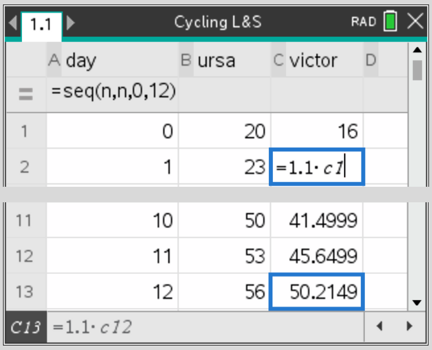
***Note:** The cell reference **b1** is relative to the cell location. When filled down, it renews to '**=b2+3**', '**=b3+3**' etc.*



(b)(ii) To generate the sequence, $v_0 = 16, v_{n+1} = 1.1 \times v_n$:

- In cell **C1**, enter **16**, as shown.
- In cell **C2**, enter the formula, **=1.1×*c*1**, , and then press [enter].
- Return to cell **C2** and press [ctrl] [menu] > **Fill**.
- Press ▼ to cell **C13** then [enter].

***Note:** The cell reference **c1** is relative to the cell location. When filled down, it renews to, '**=1.1×c2**', '**=1.1×c3**' etc.*



Answer. On day 13, Ursa and Victor cycle 56km and 50.2km.

1.2.2. Graphical and tabular displays of sequences

Using sequence graphing and tabulation with an explicit rule in the Graphs application

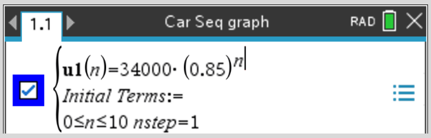
The terms of the sequence with explicit rule $u_n = 34000 \times (0.85)^n$, $n = \{0,1,\ldots,10\}$ were generated in a previous example to model the value of an asset after n years.

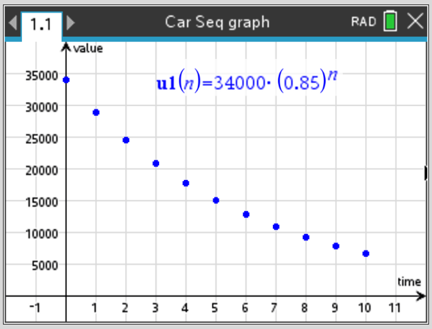
(a) Use sequence graphing in the **Graphs** application to plot the sequence. Hence graph a continuous function that models this sequence.

(b) Display the sequence as a table of values and determine the value of the asset after 6 years (to the nearest dollar).

(a) To create a **Sequence** graph with a rule $u_n = 34000 \times (0.85)^n$, $n = \{0,1,\ldots,10\}$, on a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Sequence** > **Sequence**.
- Enter $\mathbf{u1(n) = 34000 \times 0.85^n}$, Initial Terms:= (blank), $\mathbf{0 \le n \le 10}$ **nstep=1**, as shown.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: **–2** XMax: **12** XScale: **1**
  YMin: **–5000** YMax: **40000** YScale: **5000**

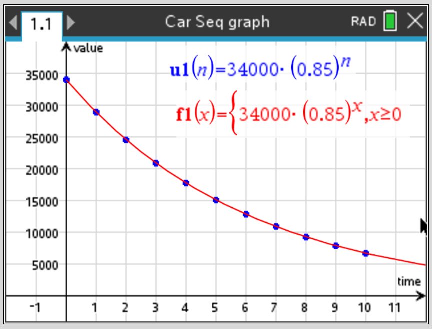




***Note: Sequence** graph accepts either the explicit rule,* $\mathbf{u1(n) = 34000 \times 0.85^n}$*, or the recurrence relation* $u_{n+1} = 0.85 \times u_n$, $u_0 = 34000$ *. However, note that the required notation is:*
$\mathbf{u1(n) = 0.85 \times u1(n-1)}$*, Initial Terms:=**34000**.*

To add a **Function** graph that models the sequence:

- Press [menu] > **Graph Entry/Edit** > **Function**.
- Enter $\mathbf{f1(x) = 34000 \times 0.85^x \mid x \ge 0}$ ('|' via [ctrl] [=]).
- Press [ctrl] [menu] > **Hide/Show** > **Show Lined Grid**.
- Show multiple axes labels by hovering over an axis and pressing [ctrl] [menu] > **Attributes**. In the popup menu that follows, use ▼ to select the **Tick Labels** attribute, then press ◀ to select **Multiple Labels**.
- Rename the axes labels. Click the label ***x*** (by pressing [click] twice) and change it to **time**. Similarly, change the label ***y*** to **value**.

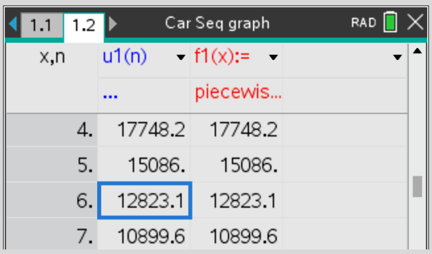


(b) To display the sequence as a table of values:

- Press [ctrl] [T] to toggle the table on/off (alternatively, press [menu] > **Table** > **Split-screen Table**).

To display the table on a separate page:

- Press [doc▾] > **Page Layout** > **Ungroup**.



Answer. After 6 years, the value of the asset is \$12,823 to the nearest dollar.

Generating and displaying values of a sequence in the Lists & Spreadsheet application

Suppose that a theatre has 19 rows of seats. There are 24 seats in the first row, and each row after the first has 3 more seats than the previous row.

Ticket prices are $60 for rows 1 to 6, $50 for rows 7 to 12 and $45 for rows 13 to 19.

(a) Write a recurrence formula for the number of seats and use the **Lists & Spreadsheet** application to determine the number of seats in each row.

Use the sequence generated in **part (a)** above to find the following:

(b) The total number of seats in the theatre.

(c) The total income from ticket sales for a sold-out show.

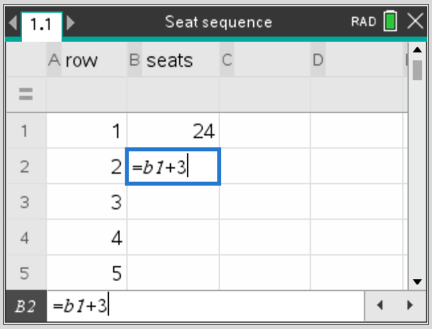
(a) The recurrence relation is $u_{n+1} = u_n + 3,\ u_1 = u_0 + 3 = 24$

To set up the spreadsheet, on a **Lists & Spreadsheet** page:

- In the heading cells, enter **row** and **seats**, as shown.

To recursively generate the sequence of theatre row numbers:

- In cells **A1** and **A2** enter **1** and **2** respectively. Navigate to cell **A1** then press [⇧shift] ▼ to select both cells **A1** and **A2**.
- Press [ctrl] [menu] > **Fill**. Press ▼ to cell **A19** then [enter].



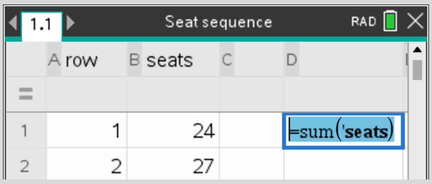
To recursively generate the number of seats in each row:

- In cell **B1** enter **24**. In cell **B2** enter the formula **=*b1*+3**.
- Navigate to cell **B2**, press [ctrl] [menu] > **Fill**. Press ▼ to cell **B19** then [enter].

Note: *The cell reference above is relative to the cell location. When filled down, it renews to '**=b2+3**', '**=b3+3**' etc.*

(b) To find the total number of seats in the theatre:

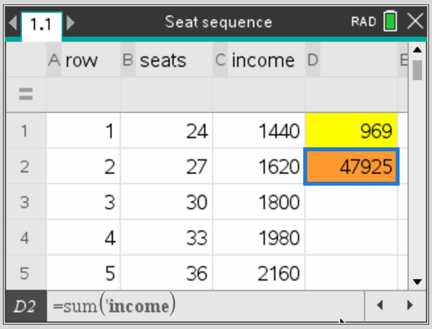
- Navigate to cell **D1**, press [menu] > **Data** > **List Maths** > **Sum of Elements**. Press [var] select **seats** then [enter].



Answer: Total number of seats in the theatre is 969.

(c) To find the income from each row of seats:

- In the **column C** heading cell enter the heading **income**.
- In cell **C1** enter the formula **=*b1*×60**. Navigate to cell **C1**, press [ctrl] [menu] > **Fill**. Press ▼ to cell **C6** then [enter].
- In cell **C7** enter the formula **=*b7*×50**. Navigate to cell **C7**, press [ctrl] [menu] > **Fill**. Press ▼ to cell **C12** then [enter].
- In cell **C13** enter the formula **=*b13*×45**. Navigate to cell **C13**, press [ctrl] [menu] > **Fill**. Press ▼ to cell **C19** then [enter].



To find the total income from ticket sales for a sold-out show:

- Navigate to cell **D2**, press [menu] > **Data** > **List Maths** > **Sum of Elements**. Press [var] select **income'** then [enter].

Answer: Total income for a sold-out show total is $47,925.

Note: *Shortcuts to navigate the **Lists & Spreadsheet** application.*

[ctrl] [1]: *Move to end of list/page down* [ctrl] [7]: *Move to start of list/page up*

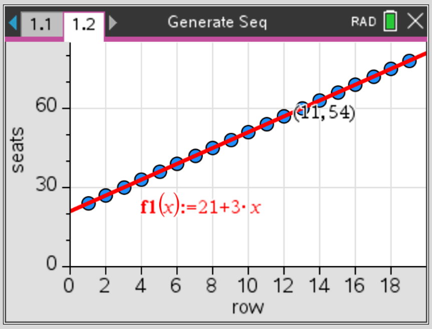
[ctrl] [3]: *Page down* [ctrl] [9]: *Page up* [ctrl] [G]: *Go to cell (enter the cell reference)*

Graphing a sequence in the Data & Statistics application

In the previous problem, the sequences with the headings ‘row’ and ‘seats’ were generated in the **Lists & Spreadsheet** application. Use these lists to graph the number of seats against the row number. Hence graph a continuous function that models this situation.

To graph the number of seats against the row number, add a **Data & Statistics** page by pressing [ctrl] [I] or [ctrl] [doc▾], then:

- Press [tab] and select **row**, then press [tab] and select **seats**.
- Press [menu] > **Window/Zoom** > **Window settings**.
  In the dialog box that follows, enter the following values:
  XMin: **0** XMax: **20** YMin: **0** YMax: **85**



To graph a continuous function that models this situation:

- Press [menu] > **Analyse** > **Plot Function**
- Enter $f1(x) := 21 + 3x$

1.2.3. Financial and practical applications of sequences

Modelling the effect of inflation using a recurrence relation

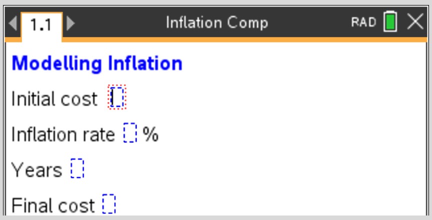
A public housing authority increases the rent on its properties once per year in line with the annual inflation rate. Assume that the annual inflation rate remains steady at r % per year. If the rent on a property is u_0 at the start of a lease, then the weekly rent after n years is modelled by the recurrence relation $u_{n+1} = \left(1 + \frac{r}{100}\right) \times u_n, u_0 = \text{initial rent}$.

(a) Use the recurrence relation to set up an interactive inflation calculator that can be edited with inputs u_0, r and time. Hence find the rent after 10 years if the initial rent, u_0, is \$360 and $r = 4\%$.

(b) Create an interactive graphical display of the effect of different inflation rates on an initial rental value. If $u_0 = 320$, compare the rents after 10 years and 20 years for r = 2, 4, 6 and 8%.

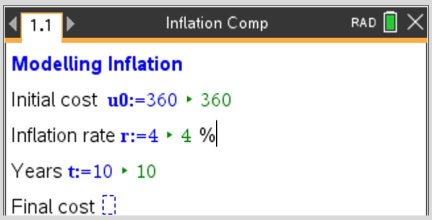
(a) To set up inputs u_0, r and time, on a **Notes** page:

- Enter the text captions, **Initial cost** etc., as shown.
- Place the cursor to the right of the caption, **Initial cost** then insert a **Maths Box** by pressing [ctrl] [M].
- Similarly, insert **Maths Boxes** next to the other captions.



To assign input values, $u_0 = 360, r = 4$, time = 10 years:

- In the top Maths Box, key in ***u*0:=360** by pressing [ctrl] [⊞] (:=) for the **assign** symbol, then press [enter] to activate.
- Similarly, enter ***r*:=4** and ***t*:=10**, as shown. Leave the last **Maths Box** empty for now.

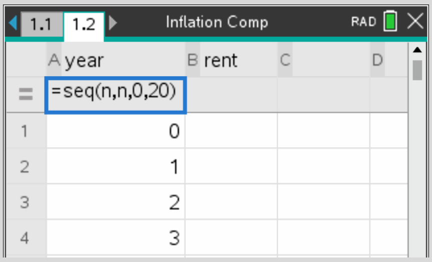


... continued

Modelling the effect of inflation using a recurrence relation (continued)

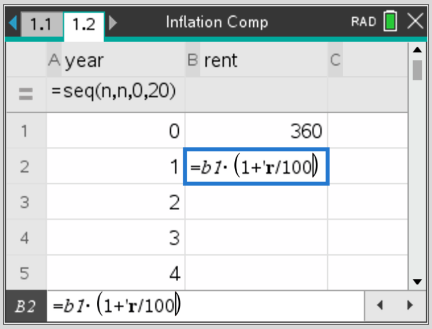
To set up a table displaying the rent after *n* years, add a **Lists & Spreadsheet** page, then:

- In the heading cells, enter **year** and **rent**, as shown.
- Navigate to the column **A** formula cell (second from the top). Enter the formula, **=seq(*n*,*n*,0,20)** by pressing [📖] [S] and navigating to **seq**. Note the syntax is **seq(Expr, Var, Low, High [,Step])**. The default **Step = 1**.



To recursively generate the terms of the sequence with recurrence relation $u_{n+1} = (1 + r/100) \times u_n, u_0 = \ldots$:

- In cell **B1**, enter the formula, **=*u*0** (press var to select ***u*0**.)
- In cell **B2**, enter **=*b*1×(1+*r*/100)** (press var to select ***r***.)
- Navigate to cell **B2**, press ctrl menu > **Fill**. Press ▼ to cell **B21** then enter.



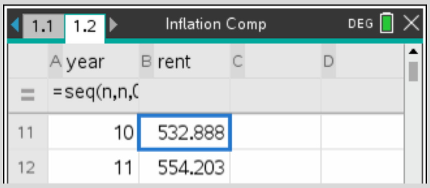
__Note:__ The cell reference __b1__ is relative to the cell location. When filled down, it renews to '__=b2×(1+r/100)__', etc.

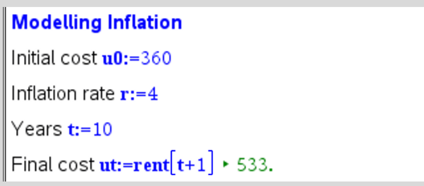
To determine the rent after 10 years if the initial rent is \$360 and the inflation rate is 4% per year:

- Navigate to cell **B11** and observe the value in the cell.

To capture the value, on the **Notes** page 1.1:

- Enter ***ut*:=rent[*t*+1]** in the last **Maths Box** (press var to select **rent** , ***t***). The (***t*+1**)th element of **rent** is assigned ***ut***.



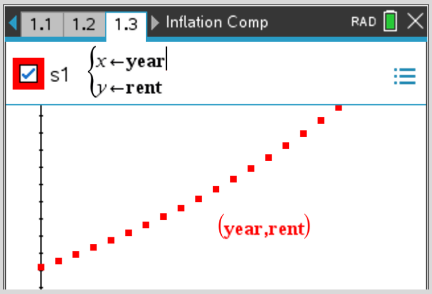


__Note:__ To show __ut__ as an integer, select the Maths Box, press ctrl menu *> __Maths Box Attributes__ > __Display Digits: Fix 0__.*

Answer. If $u_0 = 360$, after 10 years the rent will be \$533.

(b) To create a dynamic graphical display for various inflation rates, add a **Graphs** page, then:

- Press menu > **Graph Entry/Edit** > **Scatter Plot**.
- For **S1**, to enter the ***x*** variable,press var then select **year**.
- Enter the ***y*** variable by pressing var then selecting **rent**.
- Press menu > **Window/Zoom** > **Zoom-Fit**.



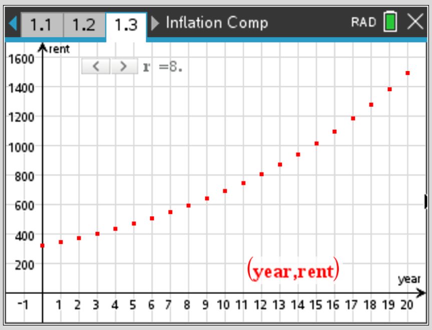
__Note:__ The graph will be renewed if the value of __u0__ is edited to __u0:=320__ in the __Maths Box__ next to the caption, __Initial cost__.

To control the value of ***r*** interactively with a slider:

- Press menu > **Actions** > **Insert Slider**.

In the dialog box that follows, enter the following values:
Variable: ***r***, Value: **8**, Minimum: **0**, Maximum**: 10**, Step Size: **1**

- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: **–2**, XMax: **21**, XScale: **1**
  YMin: **–200**, YMax: **1700**, YScale: **200**



... continued

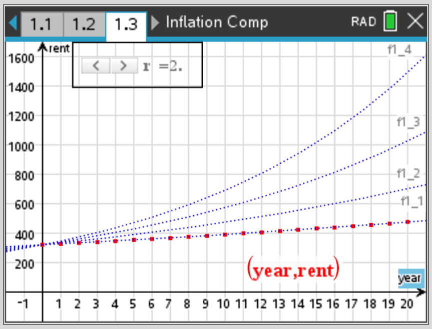
Modelling the effect of inflation using a recurrence relation (continued)

- Press ctrl menu > **Hide/Show** > **Show Lined Grid**.
- Show multiple axes labels by hovering over an axis and pressing ctrl menu > **Attributes**. In the popup menu that follows, use ▼ to select the **Tick Labels** attribute, then press ◀ to select **Multiple Labels**. Press enter to save this attribute.

Rename the axes labels. Click ( ) the label ***x*** and edit it to **year**. Similarly, edit the label ***y*** to **rent**.

To graph continuous functions that model this situation:

- Press menu > **Graph Entry/Edit** > **Function**.
- Enter $\boldsymbol{f1(x) = u0 \times (1 + r/100)^x \mid r = \{2,4,6,8\}}$
- Hover over a continuous graph, press ctrl menu > **Attributes** and select **Line style is dotted**.



Answer. If the initial rent is \$320, for r = {2%,4%,6%,8%}, after 10 years the final rent is {\$390, \$474, \$573, \$691}, and after 20 years, {\$476, \$701, \$1,026, \$1,492}.

Comparing simple and compound interest loans using explicit rules

A lender offers two reverse mortgage loan options, which do not require the borrower to make repayments until the property is sold. The *Simple Loan* option charges simple interest of 12% per year on the loan principal, calculated at the end of each year. The *Low Loan* option charges interest of 8% per year, compounded at the end of each year.

The explicit rules for the value of the debt after n years is given by:

$$\textit{Simple loan:}\ u_n = u_0 \times (1 + 0.12n) \qquad \textit{Low Loan:}\ v_n = v_0 \times (1.08)^n$$

Consider a loan principal of \$20,000.

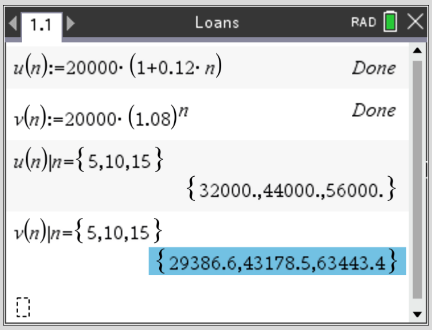
(a) Compare the value of the debts for the two options after 5 years, 10 years and 15 years.
(b) For each option, find the time taken for the debt to exceed \$50,000.
(c) For the two options, create graphical and tabular displays of the debt for the first 15 years of the loans. Hence determine when the debts for both loans are approximately equal.

(a) To assign the rules as $u(n)$ and $v(n)$, on a **Calculator** page:

- Enter $\boldsymbol{u(n) := 20000(1 + 0.12n)}$ by pressing ctrl (:=) to key in the **assign** symbol. Similarly:
- Enter $\boldsymbol{v(n) := 20000(1.08)^n}$.

To find the debts after 5, 10 and 15 years:

- Enter $\boldsymbol{u(n) \mid n = \{5,10,15\}}$, then $\boldsymbol{v(n) \mid n = \{5,10,15\}}$.



Answer. *Simple Loan*: \$32,000, \$44,000, \$56,000
Low Loan: \$29,386.60, \$43,178.50, \$63.443.40

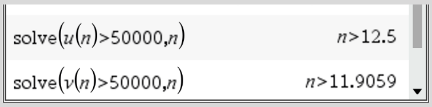
***Note**: The '|' symbol can be found via* ctrl =.

... continued

Comparing simple and compound interest loans using explicit rules (continued)

(b) To find the time for the debts to exceed $50,000:

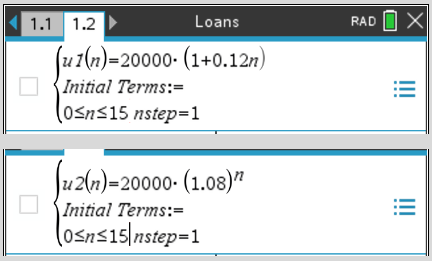
- Press [menu] > **Algebra** > **Solve**, enter **solve(*u*(*n*) > 50000,*n*)**.
- Similarly, enter **solve(*v*(*n*) > 50000,*n*)**.
- Enter $\boldsymbol{u(n)\,|\,n=\{5,10,15\}}$, then $\boldsymbol{v(n)\,|\,n=\{5,10,15\}}$.



Answer. *Simple Loan*: exceeds $50,000 after 13 years
Low Loan: exceeds $50,000 after 12 years

(c) To graph the sequences u_n and v_n, on a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Sequence** > **Sequence**.
- Enter as shown with $\boldsymbol{u1(n)=20000(1+0.12n)}$ for u_n.
- Press the ▼ key and enter $\boldsymbol{u2(n)=20000\cdot(1.08)^n}$ for v_n.
- In each case, enter $\boldsymbol{0\le n\le 15}$ *nstep* $\boldsymbol{=1}$, as shown.
  The field '*Initial Terms*' is not required for explicit rules.



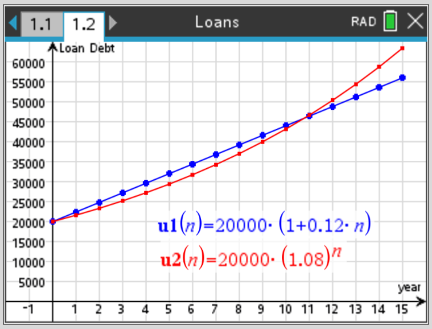
***Note**: You can also enter* $\boldsymbol{u1(n)=u(n)}$ *and* $\boldsymbol{u2(n)=v(n)}$ *since **u(n)** and **v(n)** are already defined.*

To adjust the window settings and add some enhancements:

- Press [menu] > **Window/Zoom** > **Window settings**.
  In the dialog box that follows, enter the following values:
  XMin: **–2**, XMax: **16**, XScale: **1**
  YMin: **–5000**, YMax: **65000**, YScale: **5000**

To connect the plot points (optional), hover over a plot point:

- Press [ctrl] [menu] > **Attributes** > **Points are connected**.
- Press [ctrl] [menu] > **Hide/Show** > **Show Lined Grid**.
- Show multiple axes labels by hovering over an axis and pressing [ctrl] [menu] > **Attributes**. In the popup menu that follows, use ▼ to select the **Tick Labels** attribute, then press ◀ to select **Multiple Labels**. Press [enter] to save this attribute.
- Rename the axes labels. Click twice (🖱) on the label ***x*** and change it to **year**. Similarly, edit the label ***y*** to **Loan Debt**.

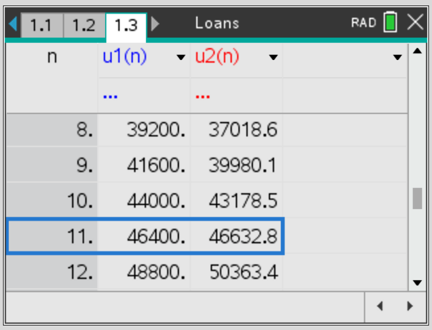


(c) To obtain a tabular display of the sequences u_n and v_n, from the **Graphs** page above:

- Press [ctrl] [T]

To view the table of values on a separate page:

- Press [doc▾] > **Page Layout** > **Ungroup**.
- Navigate to page 1.3 to view the table of values.



Answer: The graphs and the table confirm that the loan debts are equal at the outset and approximately equal after 11 years, when the difference is approximately
$46,633 – $46,400 = $233.

Analysing a sequence and the cumulative sum of its terms using a spreadsheet

Suppose that members of a school's book club agree to read the *Lord of the Rings* trilogy before their next monthly meeting. Farah is a club member who plans to read the 1501 pages of the trilogy by reading 20 pages on the first day, and each subsequent day reading 5 pages more than the previous day. The number of pages that will be read on day n is given by the recurrence relation $u_{n+1} = u_n + 5,\ u_1 = 20$, where $n = 1, 2, 3, \ldots$(and if required, $u_0 = u_1 - 5$).

Note: *The following is intended to illustrate the use of some functionalities of the technology that may be useful in modelling, problem solving and mathematical investigation.*

Use the **Lists & Spreadsheet** application to create a tabular display of the following.

(a) The number of pages read on day *n*.
(b) The cumulative number of pages read by the end of day *n*.
(c) The percentage of the novel that has been read by the end of day *n*.

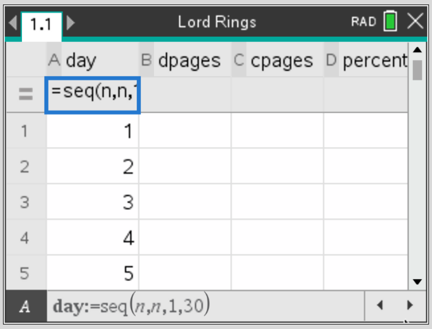
To set up the spreadsheet, on a **Lists & Spreadsheet** page:

- In the column A, B, C and D heading cells respectively, enter the headings **day**, **dpages**, **cpages** and **percent**.

To generate the sequence of days, $n \in \{1, 2, \ldots, 30\}$:

- Navigate to the column A formula cell (second from the top) and enter the formula **=seq(*n*,*n*,1,30)**.

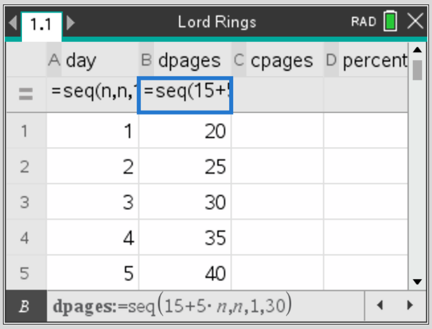
Note: *For 'seq', press* [📖][S] *and navigate to* ***seq****. The syntax is* ***seq(Expr, Var, Low, High [,Step])****. Default Step = 1.*



(a) To apply the explicit rule, $u_n = u_0 + 5n = (20 - 5) + 5n$ and generate the sequence of the number of pages read on day *n*:

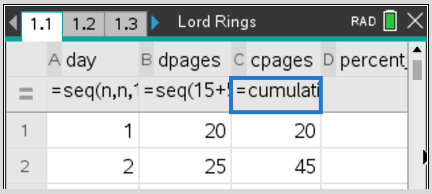
- Navigate to the column B formula cell and enter the formula **=seq(15+5*n*,*n*,1,30)**.

Note: *An alternative command is 'seqn', with formula* ***=seqn(15+5n,30)****. The syntax for an explicit rule with variable n, starting with n=1 is*
seqn(Expr(n), nMax [, CeilingValue])
For 'seqn', press [📖][S] *and navigate to* ***seqn****.*



(b) To generate a sequence of the cumulative number of pages read at the end of day *n*:

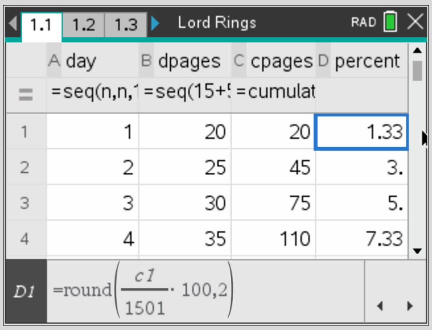
- In the column C formula cell, enter the formula **=cumulativeSum('dpages)** by pressing [📖][D] to select **cumulativeSum**, then [var] to select the stored list **dpages**.



(c) To generate a sequence of the percentage of the novel read by the end of day *n*, correct to two decimal places:

- In cell D1, enter the formula **=round(*c1*/1501×100,2)**. To select '**round**', press [📖][R] and navigate to **round**.
- Navigate to cell D1, press [ctrl][menu] > **Fill**. Press ▼ to cell C30 then [enter]. This fills the formula down to cell C30.

Note: *The cell reference* ***c1*** *is relative to the cell location. When filled down, it renews to '****=round(c2/1501×100,2)****' etc.*



Answer. The completed spreadsheet contains a tabular display of the first 30 terms of the required sequences.

Analysing a sequence and its cumulative sum as scatterplots

In the previous problem, the number of pages of the *Lord of the Rings* trilogy that Farah will read on day n, $n = 1, 2, 3, \ldots$, is given by the recurrence relation $u_{n+1} = u_n + 5,\ u_1 = 20$.

(a) Use the lists created in the previous problem to produce graphical displays of the number of pages read on day n, and the cumulative number of pages by the end of day n.

(b) Show that the explicit rule for the cumulative number of pages read by the end of day n is given by $S_n = \frac{n}{2}\left((2u_1 - 5) + 5n\right)$.

(c) Parts 1,2 and 3 of the trilogy have 529 pages, 442 pages and 530 pages, respectively. If Farah adheres to the plan, use the graphs from part (a) above to determine on which day, n, she will finish reading each of the three parts of the trilogy.

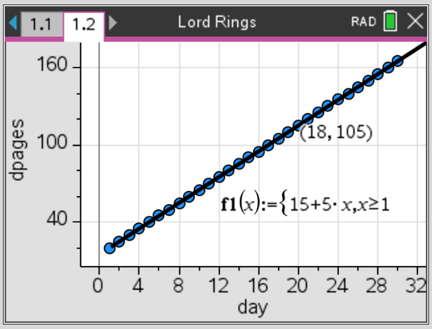
Approach 1: Using the Data & Statistics application

(a) To create a graph of the number of pages read on day n, add a **Data & Statistics** page to the previous problem, then:

- Press [tab] and select the list **day** for the horizontal axis.
- Press [tab] and select the list **dpages** for the vertical axis.

To add a continuous function that models the data and has a rule analogous to the explicit rule of the sequence:

- Press [menu] > **Analyse** > **Plot Function**.
- In the textbox that appears, enter ***f*1(*x*):=15+5*x*|*x*≥1**.



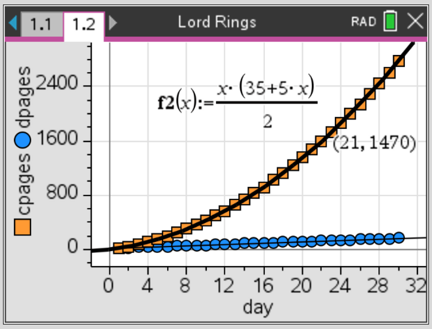
To create a graph of the cumulative number of pages read by the end of day n, on the **Data & Statistics** page:

- Press [menu] > **Plot Properties** > **Add Y variable**.
- Select list **cpages** on the vertical axis.

(b) To show that the cumulative number of pages is given by

$$S_n = n\left((2u_1 - 5) + 5n\right)/2:$$

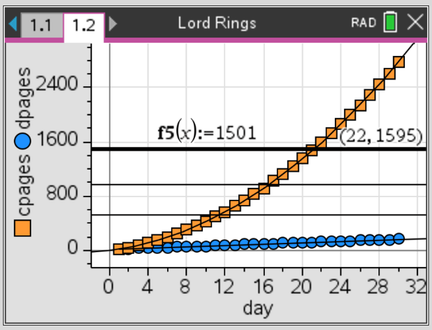
- Press [menu] > **Analyse** > **Plot Function**.
- In the textbox that appears, enter $\boldsymbol{f2(x) := x \cdot (35 + 5x)/2}$.



Answer. It is shown that the graph of ***f*2** contains all the values of the sequence list '**cpages**'.

(c) To determine graphically the day, n, when Farah will finish reading each of the three parts of the trilogy:

- Press [menu] > **Analyse** > **Plot Function** and enter $\boldsymbol{f3(x) := 529}$, $\boldsymbol{f4(x) := (529 + 442)}$ and $\boldsymbol{f5(x) := 1501}$.
- Hover over the points where each of the above function graphs intersect the **cpages** plot to read the values of n.



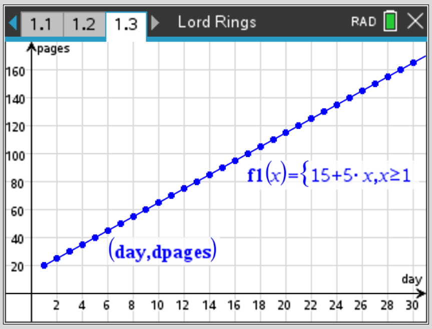
Answer. Parts 1, 2 and 3 will be finished during days 12, 17 and 22 respectively. The tabular display confirms the results.

Analysing a sequence and its cumulative sum as scatterplots (continued)

Approach 2: Using the Graphs application

(a) To create a graph of the number of pages read on day *n*:

- Add a **Graphs** page to the previous problem by pressing ctrl doc▾ (+page) > **Add Graphs**.
- Press menu > **Graphs Entry/Edit** > **Scatterplot**.
- For **S1**, to enter the ***x*** variable, press var then select **day**.
- To enter the ***y*** variable, press var then select **dpages**.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: **–2**, XMax: **31**, XScale:**2**
  YMin: **–20**, YMax: **180**, YScale: **20**
- Press ctrl menu > **Hide/Show** > **Show Lined Grid**.
- To show multiple axes labels, hover over an axis and press ctrl menu > **Attributes**. Select **Multiple Labels**. In the popup menu that follows, use ▼ to select the **Tick Labels** attribute, then press ◀ to select Multiple Labels.
  Press enter to save this attribute.
- Rename the axes labels. Click (click key) the label ***x*** and edit it to **day**. Similarly, edit the label ***y*** to **pages**.

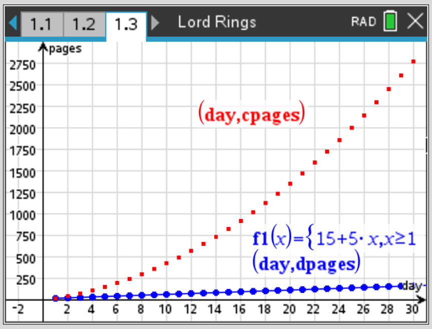


To add a continuous function that models the data and has a rule analogous to the explicit rule of the sequence:

- Press menu > **Graphs Entry/Edit** > **Function.**
- Enter ***f*1(*x*)=15+5*x*|*x*≥1**.

To add a graph of the cumulative number of pages read by the end of day *n*, on the **Graphs** page:

- Press menu > **Graphs Entry/Edit** > **Scatterplot**.
- For **S2**, to enter the ***x*** variable, press var then select **day**.
- To enter the ***y*** variable, press var then select **cpages**.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: **–3**, XMax: **31**, XScale:**2**
  YMin: **–250**, YMax: **3000**, YScale:**250**

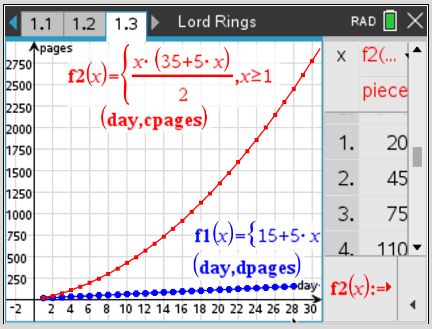


(b) To show that the cumulative number of pages is given by $S_n = n\left(\left(2u_1 - 5\right) + 5n\right)/2$:

- Press menu > **Graphs Entry/Edit** > **Function**.
- Enter $f2(x) = x \cdot (35 + 5x)/2 \mid x \geq 1$

To toggle (open and close) a table of values for function ***f*2**:

- Press ctrl T and navigate to ***f*2(*x*).** Press ctrl T to close.



Answer. It is shown that the graph of ***f*2** contains all the values of the sequence list '**cpages**'.

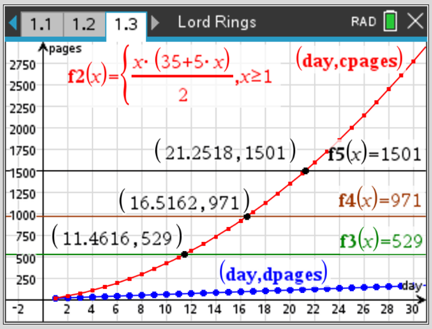
Analysing a sequence and its cumulative sum as scatterplots (continued)

Approach 2: Using the Graphs application (continued)

(c) To determine graphically the day, *n*, when Farah will finish reading each of the three parts of the trilogy:

- Press menu > **Graphs Entry/Edit** > **Function** and enter $f3(x)=529$, $f4(x)=(529+442)$ and $f5(x)=1501$.
- Press menu > **Geometry** > **Points & Lines** > **Intersection Points**. Click graph $f2$ followed by $f3$, $f4$ and $f5$.

Answer. Parts 1, 2 and 3 will be finished during days 12, 17 and 22 respectively. The tabular display confirms the results.



1.3. Linear functions, graphs, equations and models

1.3.1. Linear Functions

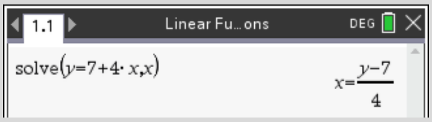
Solving a linear equation

(a) Rearrange the linear equation $y = 7 + 4x$ to make x the subject of the equation.
(b) Find the value of x when $y = 3$.
(c) Rearrange the linear literal equation $ay = b + cx$ to make x the subject of the equation.

***Note**: For the following questions, it is assumed that the TI-Nspire CX II CAS has been set to display exact answers where possible. To check, press [on] > **Settings** > **Document Settings**, and select **Calculation Mode = Auto**.*

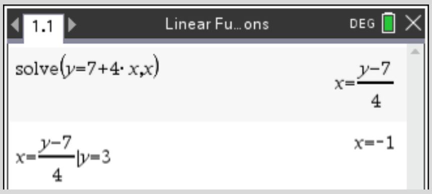
(a) On a **Calculator** page:

- Press [menu] > **Algebra** > **Solve**, and then enter **solve**$\left(y=7+4x,x\right)$.



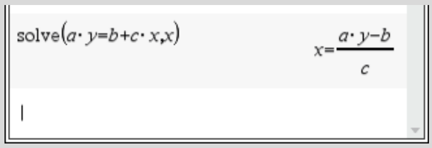
Answer: $x = \dfrac{y-7}{4}$.

(b) To find the value of x when $y = 3$:
- Press ▲ to select the previous answer and then press [enter] to copy and paste it to the editing line.
- Press [ctrl] [=] and select the | symbol, then enter $\boldsymbol{y = 3}$ as shown in the screen right.



Answer: $x = -1$.

(c) To make x the subject, use the solve command as above, but insert a multiplication symbol between the variable letters.
- Press [menu] > **Algebra** > **Solve**, and then enter **solve**$\left(a\cdot y=b+c\cdot x,x\right)$.



Answer: $x = \dfrac{ay-b}{c}$.

***Note:** When entering the product of two variables, such as the product **ay**, remember to enter the multiplication symbol between the letters, so that the calculator understands that **ay** is the product of the variables **a** and **y** (i.e. **a×y**), and not a variable with the two-letter name '**ay**'.*

Determining an equation of a straight line given two points

Find the equation of the line that passes through (2, 5) and (3, 7), using the form $y = a + bx$.

Method 1: Using the Calculator App

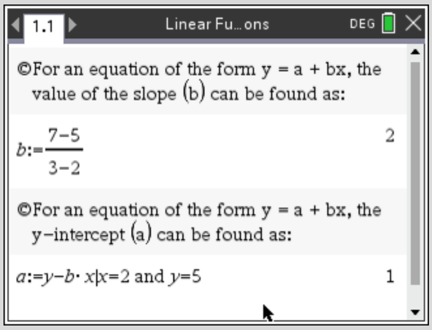
For a line with equation $y = a + bx$, for any two points (x_1, y_1) and (x_2, y_2), the values of the slope (b) and the y-intercept (a) can be calculated as follows:

$$b = \frac{y_2 - y_1}{x_2 - x_1} \text{ and } a = y_1 - bx_1$$

To use this method for the two points $(x_1, y_1) = (2,5)$ and $(x_2, y_2) = (3,7)$, on a **Calculator** page:

- Enter the command $\boldsymbol{b := \frac{7-5}{3-2}}$
- Enter the command $\boldsymbol{a := y - b \times x \mid x = 2 \text{ and } y = 5}$

Answer: The equation of the line is $y = 1 + 2x$.

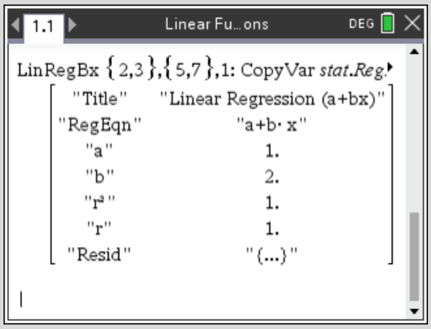


Method 2: Using Linear Regression on the Calculator App

On a **Calculator** page:

- Press menu > **Statistics** > **Stat Calculations** > **Linear Regression (a + bx)**
- For X List, enter **{2,3}**
- For Y List, enter **{5,7}**
- Press **OK** to calculate the equation of the line.

***Note**: The braces (set brackets) can be entered via* ctrl).



Method 3: Using Linear Regression on the Lists & Spreadsheet App

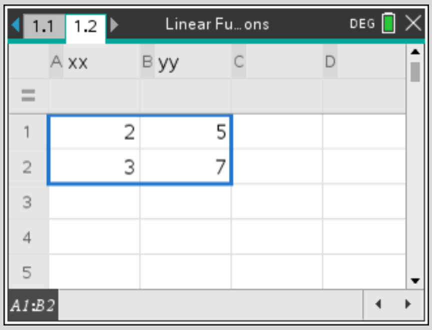
On a **Lists & Spreadsheet** page:

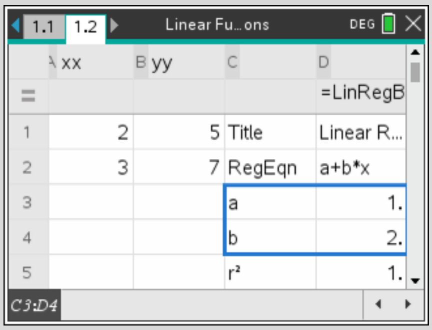
- In the column heading cells A and B, enter the variable names ***xx*** and ***yy***.

***Note:** Using the variable names **xx** and **yy** avoids any conflict with the spreadsheet column variables **x** and **y**.*

- Enter the coordinate data for the two variables into columns A and B as shown.
- Press menu > **Statistics** > **Stat Calculations** > **Linear Regression (a+bx)**
- In the dialog box that follows:
  - For 'X List', select ***xx***
  - For 'Y List', select ***yy***
- Press **OK** to calculate the equation of the line.

This displays the value of a and b in columns C and D of the spreadsheet.





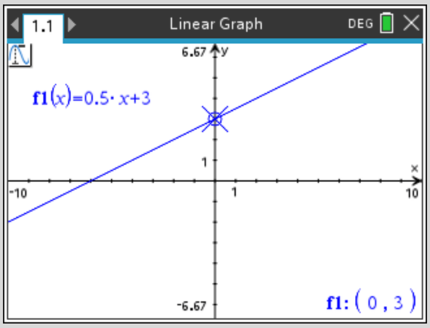
Plotting and working with linear graphs

Graphing functions and relations are usually done in the **Graphs** app. The following examples highlight the basic operations for working with linear graphs.

(a) Plot the graph of the equation $y = 0.5x + 3$ and label the axis intercepts.
(b) Using the graph, find the value of y when $x = 3$.
(c) Using the graph, find the value of x when $y = 6$.

(a) On a **Graphs** page:

- Enter $\boldsymbol{f1(x) = 0.5x + 3}$

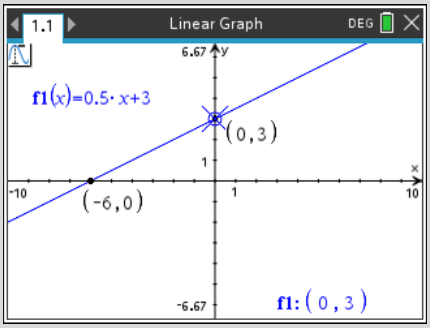


To trace coordinates along this linear graph:

- Press [menu] > **Trace** > **Graph Trace**
- Use ◀ or ▶ to move the trace cursor along the graph.
- To move the trace cursor to a point with a specific x value, enter the x value, then press [enter].

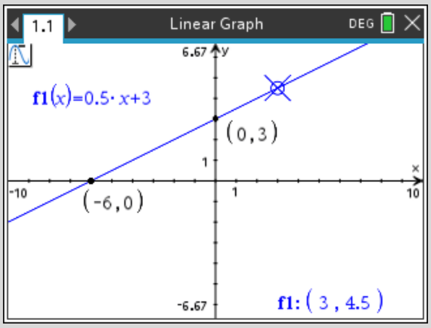
To label the axis intercepts, with **Graph Trace** on:

- Use ◀ or ▶ to move the trace cursor to the x-intercept (it will be highlighted when the cursor is at the x-intercept), then click (or press [enter]) to add the label (–6,0).
- Use ◀ or ▶ to move the trace cursor to the y-intercept (it will be highlighted when the cursor is at the y-intercept), then click (or press [enter]) to add the label (0,3).
- Press [esc] to exit **Graph Trace**.



(b) To find the value of y when $x = 3$.

- Press [menu] > **Trace** > **Graph Trace**
- Press [3] [enter] to move the trace cursor to the point on the graph where $x = 3$.



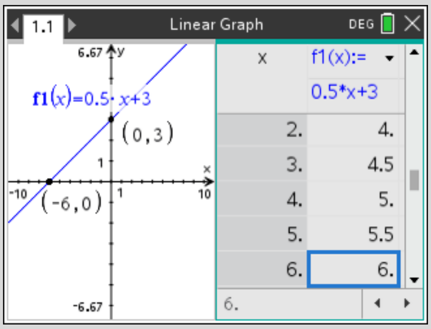
The value of y can be read from the coordinates (3, 4.5) displayed in the bottom right of the screen.

- Press [esc] to exit **Graph Trace**.

Answer: When $x = 3$, the value of $y = 4.5$.

(c) To find the value of x when $y = 6$:

- Press [ctrl] [T] to get a table of values of a structured set of the coordinates.
- Scroll through the table to find the coordinates for which $y = 6$.



The value of x can be read from the row containing the coordinates (6, 6).

- Press [ctrl] [T] to hide the table of values.

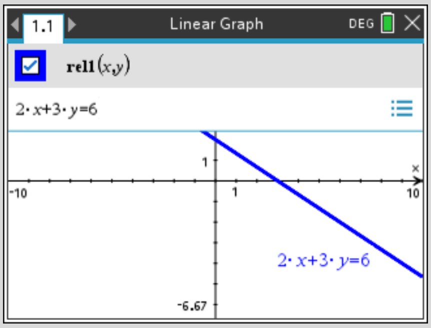
Answer: When $y = 6$, the value of $x = 6$.

Plotting a linear equation of the form ax + by = c

The graph of a linear equation expressed in the form $ax + by = c$ can be plotted with the 'relation' graphing feature. As an example, the graph of the equation $2x + 3y = 6$ will be plotted.

On a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Relation**.
- For **rel1**, enter $\mathbf{2x+3y=6}$.
- Press [menu] > **Trace** > **Graph Trace** to locate any important points.
- Use ◀ or ▶ to move the trace cursor along the graph.
- To move the trace cursor to a point with a specific x value, enter the x value, then press [enter].



Solving two linear equations simultaneously

Solve the following pair of linear equations simultaneously for x and y: $2x+3y=4$ and $x-y=3$

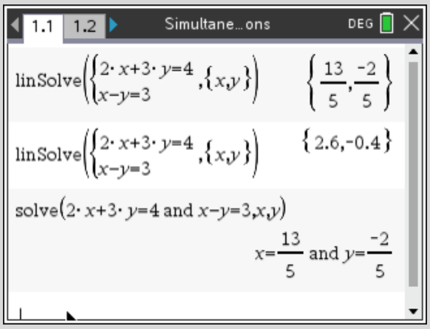
Method 1 – Solving on the Calculator App

On a **Calculator** page:

- Press [menu] > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**.
- In the dialog box that is displayed, enter the following:
  - **Number of equations = 2**
  - **Variables =** $\boldsymbol{x, y}$ (then click **OK**)
- Enter the equations into the template and press [enter], or [ctrl] [enter] to express your answers in decimal form.

Answer: $x=\dfrac{13}{5}=2.6$ and $y=\dfrac{-2}{5}=-0.4$

Note: An alternative entry method is to enter the following: $\mathbf{solve(2x+3y=4 \text{ and } x-y=3, x, y)}$ *as shown right.*



Method 2 – Solving on the Graphs App

To solve the pair of equations simultaneously for x and y, on a **Graphs** page:

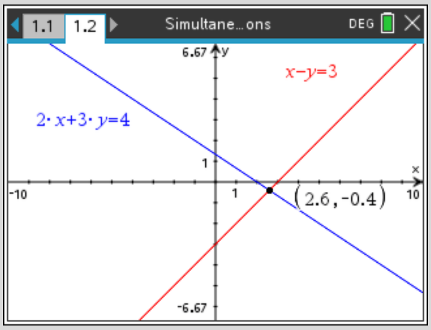
- Press [menu] > **Graph Entry/Edit** > **Relation**.
- For **rel1**, enter $\mathbf{2x+3y=4}$.
- To add **rel2**, press [ctrl] [G] and enter $\mathbf{x-y=3}$.

To find the intersection point of the two lines:

- Press [menu] > **Analyse Graph** > **Intersection**.
- Use the trackpad to position the **lower bound** to the left of the intersection point and then press [enter].
- Use the trackpad to position the **upper bound** to the right of the intersection point and then press [enter].

The coordinates of the point of intersection (2.6, –0.4) are now displayed.

- Press [esc] to exit from the **Intersection Point(s)** tool.



1.3.2. Linear Models

Modelling with a single linear function

A basic mobile phone plan designed for school students charges a flat fee of \$15 per month plus a timed call charge of 13 cents per minute. Text messaging is free.

(a) If the time spent on calls per month for students varies from 0 to 120 minutes, construct an equation that determines the monthly cost, y dollars, for time (x minutes) spent making calls per month, and graph this equation in a suitable window.

(b) Calculate the monthly cost for multiples of 10 minutes in the time making calls.

(a) Make sure all variables are in the same units so if the flat fee = \$15, then the timed call charge should be expressed as \$0.13 per minute.

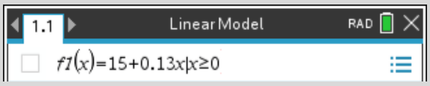
If x = *time* (in minutes) and y = *monthly cost* (in dollars) and so the following equation can be constructed:

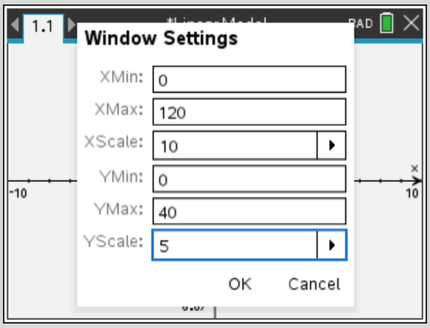
$$y = 15 + 0.13x \text{ (where } x \geq 0\text{)}.$$

To view a graph of this equation, on a **Graphs** page:

- Enter the equation $\boldsymbol{f1(x) = 15 + 0.13x \mid x \geq 0}$

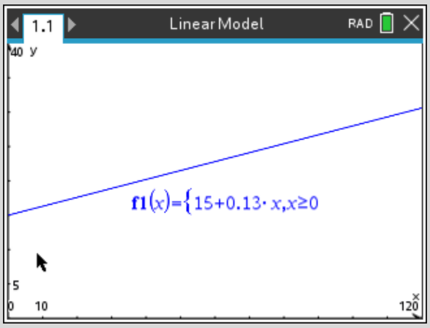
Note: The symbols ' | ' and ' ≥ ' can be found via ctrl =.





The graph is not visible in the standard viewing window. To obtain a better viewing window for this graph:

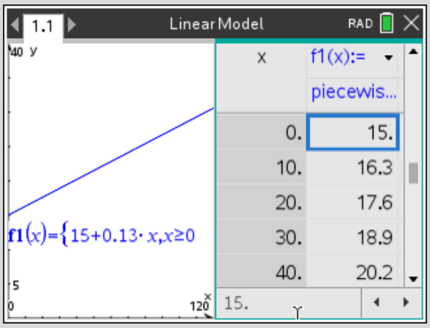
- **Press** menu **> Window/Zoom > Window Settings**
  In the dialog box that follows, enter:
  XMin = 0 XMax = 120 Xscale = 10
  YMin = 0 YMax = 40 Yscale = 5
- Press enter to view the graph in a suitable window.

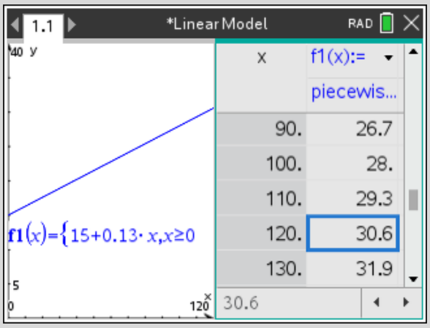


Note: *It is sometimes more efficient to use **Zoom-Fit** in such circumstances. To do this, set the values of **XMin** and **XMax** to suit the context of the model (in this example from 0 to 120 minutes), and then press* menu *> **Window/Zoom** > **Zoom-Fit.** The **Zoom-Fit** command will adjust the values of **YMin** and **Ymax** so that the graph(s) will be displayed.*

(b) To construct a table of values for the monthly cost for multiples of 10 minutes in the time making calls:

- Press ctrl T to split the window and display a table on the right half of the screen.
- Press menu > **Table** > **Edit Table Settings** and change the value of **Table Step = 10** (minutes)





Answer: As shown in the table right, the monthly cost increases by \$1.30 for every additional 10 minutes of time spent making calls, up to a monthly cost of \$30.60 for 120 minutes of call time.

Note: Press ctrl T *to hide the table.*

Modelling with two linear functions

Sandi uses carrots and apples to make their special homemade fruit juice. One week they buy 5 kg of carrots and 4 kg of apples for $31.55. The next week they buy 4 kg of carrots and 3 kg of apples for $24.65.

(a) Let x be the cost in dollars of 1 kg of carrots, and let y be the cost in dollars of 1 kg of apples. Construct two equations that represent the information above.

(b) By solving simultaneous equations, determine how much Sandi spends on 1 kg each of carrots and apples.

(c) Determine the amount Sandi spends the following week when they buy 2 kg of carrots and 1.5 kg of apples. Give your answer correct to the nearest 5 cents.

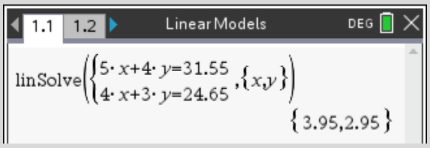
(a) Equations are $5x+4y=31.55$ and $4x+3y=24.65$.

(b) The two equations can be solved simultaneously algebraically or using graphs.

Method 1: Symbolic solving

On a **Calculator** page:

menu > **Algebra** > **Solve System of Equations** >**Solve System of Linear Equations**.

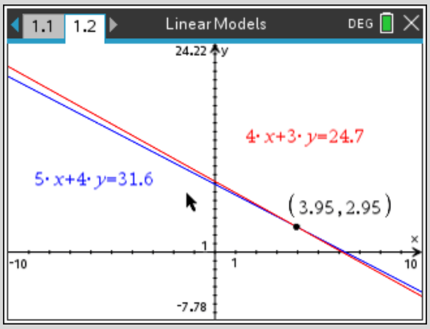


- Set the following values:
  - **Number of equations = 2**
  - **Variables =** ***x, y***
- Enter the equations into the template and press enter.

Answer: $x = \$3.95$ and $y = \$2.95$

Method 2: Graphic solving

On a **Graphs** page:



- Press menu > **Graph Entry/Edit** > **Relation**.
- For **rel1**, enter $\mathbf{5x+4y=31.55}$.
- To add **rel2**, press ctrl G and enter $\mathbf{4x+3y=24.65}$.
- Press menu > **Window/Zoom** > **Zoom-Fit**.

To find the intersection point of the two lines:

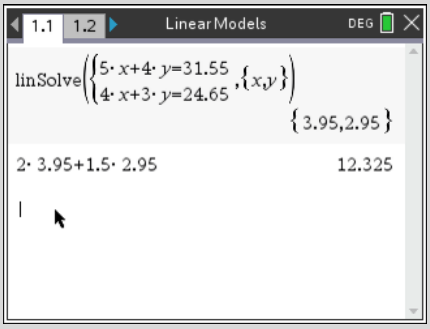
- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**.
- Click once on each line graph in turn to select them.

The coordinates of the point of intersection (3.95, 2.95) are now displayed.

- Press esc to exit from the **Intersection Point(s)** tool.

Note**: The intersection point can also be found via* menu *>* ***Analyse Graph > Intersection*.*

Answer: $x = \$3.95$ and $y = \$2.95$



(c) Use **Calculator** page to solve

$\mathbf{2\times3.95+1.5\times2.95}$ enter

Answer: 12.325 correct to nearest 5c is $12.35.

1.3.3. Piecewise linear functions

Constructing a piecewise linear function

Consider the following piecewise linear function, which is composed of three different equations, depending on the domain (that is, on the x values).

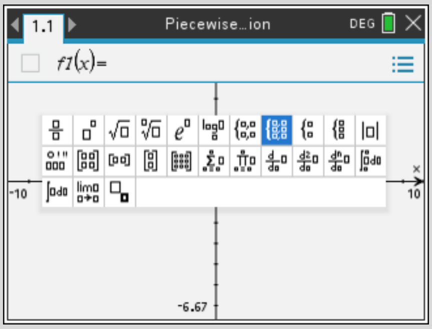
$$\begin{aligned} y &= x-3, & x &\le 1 \\ y &= 2x-4, & 1 < x &\le 4 \\ y &= x, & x &> 4 \end{aligned}$$

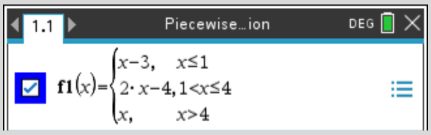
Plot the graph of this piecewise linear function and then find the value of y when $x = 7$.

On a **Graphs** page:

- Press [templates] and then select the **Piecewise Function** template as shown right.
- In the dialog box that follows, select **Number of function pieces = 3**
- In the template displayed, enter the following:

$$f1(x)=\begin{cases} x-3, & x\le 1 \\ 2x-4, & 1<x\le 4 \\ x, & x>4 \end{cases}$$



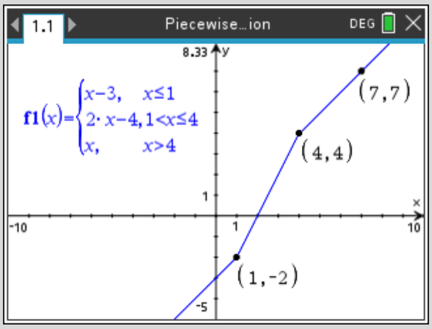


To display coordinates along the graph:

- Press [menu] > **Trace** > **Graph Trace**
- Use ◀ or ▶ to move the trace cursor along the graph.

With **Graph Trace** on, it is possible to place coordinate labels at the points where the straight-line sections join (at $x = 1$ and $x = 4$):

- Press [1] [enter] to move the trace cursor to the point on the graph where $x = 1$, then press [enter] to display the coordinates of that point.
- Press [4] [enter] to move the trace cursor to the point on the graph where $x = 4$, then press [enter] to display the coordinates of that point.

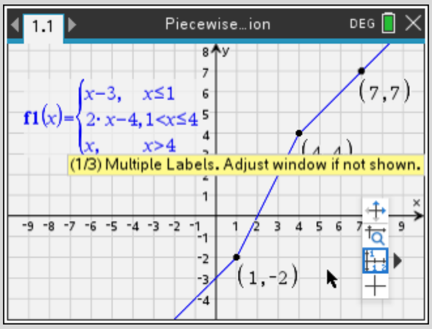


To find the value of y when $x = 7$ (with **Graph Trace** on):

- Press [7] [enter] to move the trace cursor to the point on the graph where $x = 7$, then press [enter] to display the coordinates of that point.

Answer: $y = 7$

***Note:** The screen shown right has been enhanced to display a lined grid (via [menu] > **View** > **Grid** > **Lined Grid**). Also, the axis values are labelled, which can be displayed by hovering over one of the axes, then pressing [ctrl] [menu] > **Attributes**. In the popup menu that follows, use ▼ to select the **Tick Labels** attribute, then press ◀ to select Multiple Labels. Press [enter] to save this attribute.*



Constructing a step graph

The following shows the car parking fees in a shopping carpark for different time intervals.

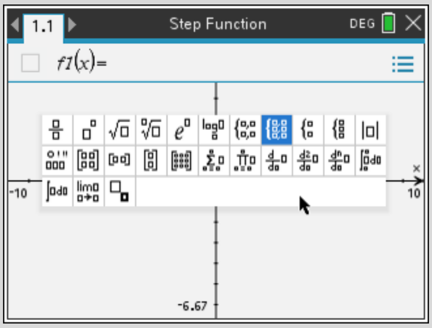
- 0−2 hours $1.00
- 2−4 hours $2.50
- 4−6 hours $5.00
- 6+ hours $6.00

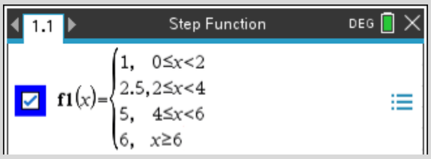
Construct a step graph to represent this information.

To draw line segments to represent each interval, on a **Graphs** page:

- Press `template` and then select the **Piecewise Function** template as shown right.
- In the dialog box that follows, select **Number of function pieces = 4**
- In the template displayed, enter the following:

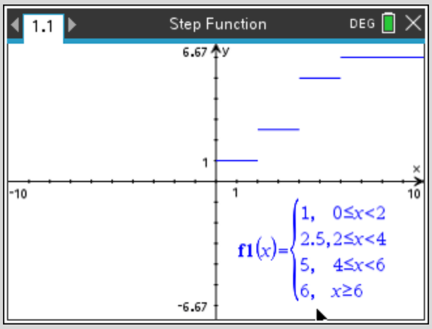
$$f1(x)=\begin{cases} 1, & 0 \le x < 2 \\ 2.5, & 2 \le x < 4 \\ 5, & 4 \le x < 6 \\ 6, & x \ge 6 \end{cases}$$





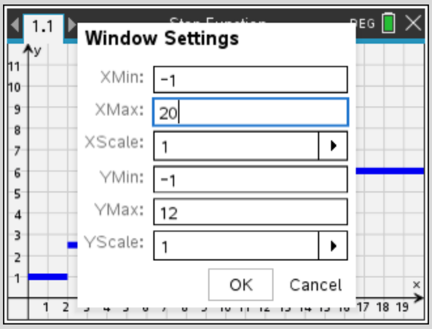
To display coordinates along the graph:

- Press `menu` **> Trace > Graph Trace**
- Use ◀ or ▶ to move the trace cursor along the graph.

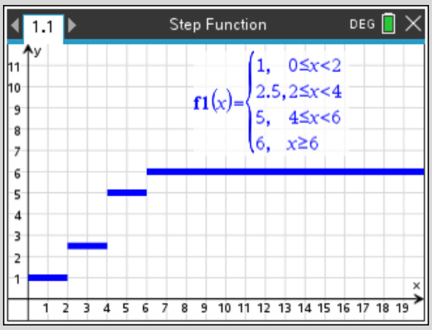


A more suitable window for viewing the graph can be obtained in the following manner:

- **Press** `menu` **> Window/Zoom > Window Settings**
  In the dialog box that follows, enter:
  Xmin = –1 XMax = 20 Xscale = 1
  Ymin = –1 Ymax = 12 Yscale = 1
- Press `enter` to view the graph in the changed window.



Notes:
1. *The screen shown right has been enhanced to display a lined grid (via* `menu` ***> View > Grid > Lined Grid****).*
Also, the axis values are labelled, which can be displayed by hovering over one of the axes, then pressing `ctrl` `menu` *>* ***Attributes****. In the popup menu that follows, press ▼ to select the* ***Tick Labels*** *attribute, then press ◀ to select Multiple Labels. Press* `enter` *to save this attribute.*
2. *Graph attributes can also be changed, by hovering over the graph, then pressing* `ctrl` `menu` *>* ***Attributes****. As an example, the first attribute is* ***Line Weight****, which can be changed to make the line segments thicker (as shown right).*



1.4. Matrices

1.4.1. Matrix definitions and types

Understanding the matrix definition and notation

A matrix is a rectangular array of elements.

The order of a matrix is $m \times n$, where m is the number of rows and n is the number of columns.

Using matrices to store and display information

Matrices are used to store and display information that can be presented in a rectangular array of rows and columns such as databases and links in social networks and road networks.

Kristy keeps a record of her mathematics test results.

Her first four test results in General Mathematics were 95%, 86%, 98% and 81%.

Her first four test results in Mathematical Methods were 83%, 78%, 86% and 83%.

Express Kristy's mathematics test results in the form of a 2×4 matrix *T*.

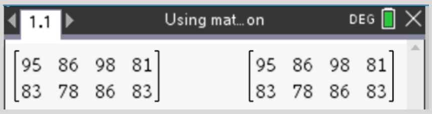
Matrix *T* is required to have order 2×4.

This means 2 rows and 4 columns.

$$T = \begin{bmatrix} 95 & 86 & 98 & 81 \\ 83 & 78 & 86 & 83 \end{bmatrix}$$

On a **Calculator** page, create *T* as follows:

- Press menu > **Matrix & Vector** > **Create** > **Matrix**.
- Set the number of rows to be 2 and the number of columns to be 4.
- Enter as shown.



***Note**: Alternatively, to create a 2 x 4 matrix, press* [book key] 5*, select the **m-by-n Matrix** template and complete as above. There is also a templates key* [templates key] *which will display a similar screen.*

Creating different types of matrices

A row matrix has only one row (order $1 \times n$).

(a) Create the matrix $\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$.

A column matrix has only one column (order $m \times 1$).

(b) Create the matrix $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

A square matrix has the same number of rows and columns (order $n \times n$).

(c) Create the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

A zero matrix has all of its elements equal to zero.

(d) Create the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

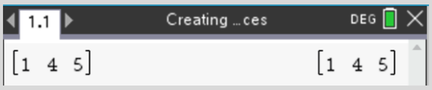
An identity matrix is a square matrix that has the number 1 for all elements on the leading (main) diagonal and 0 for all other elements.

(e) Create the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$ is a 1×3 matrix i.e. 1 row and 3 columns.

On a **Calculator** page, create the matrix as follows:

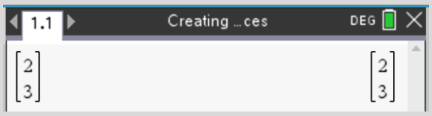
- Press [book] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 1 and the number of columns to be 3 (the [tab] key can be used to go from row selection to column selection)
- Enter as shown.



(b) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a 2×1 matrix i.e. 2 rows and 1 column.

On a **Calculator** page, create the matrix as follows:

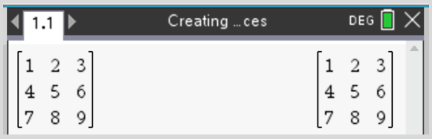
- Press [book] [5], select the **2-by-1 Matrix** template and enter as shown.



(c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a 3×3 matrix i.e. 3 rows and 3 columns.

On a **Calculator** page, create the matrix as follows:

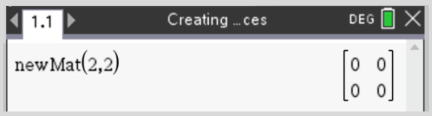
- Press [book] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 3.
- Enter as shown.



(d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 2×2 matrix i.e. 2 rows and 2 columns.

On a **Calculator** page, create the matrix as follows:

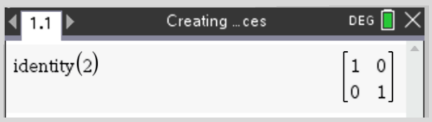
- Press [menu] > **Matrix & Vector** > **Create** > **Zero Matrix**.
- Enter as shown.



(e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a 2×2 matrix i.e. 2 rows and 2 columns.

On a **Calculator** page, create the matrix as follows:

- Press [menu] > **Matrix & Vector** > **Create** > **Identity**.
- Enter as shown.



1.4.2. Matrix arithmetic

Adding and subtracting matrices

Two matrices can be added or subtracted if they have the same order.

Addition and subtraction are performed by adding or subtracting the corresponding elements in each matrix.

In general, for 2×2 matrices, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $A \pm B = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$.

A fruit and vegetable cooperative has three shops, S_1, S_2 and S_3.

On a particular Monday:

- S_1 sold 45 avocados, 18 lettuces and 11 watermelons.
- S_2 sold 35 avocados, 18 lettuces and 9 watermelons.
- S_3 sold 47 avocados, 29 lettuces and 10 watermelons.

On a particular Tuesday:

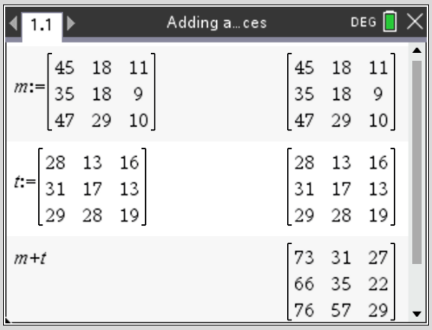
- S_1 sold 28 avocados, 13 lettuces and 16 watermelons.
- S_2 sold 31 avocados, 17 lettuces and 13 watermelons.
- S_3 sold 29 avocados, 28 lettuces and 19 watermelons.

Let the sales for the Monday be denoted by matrix *M* and the sales for the Tuesday be denoted by matrix *T*.

(a) Find the sales for the three shops over the two days.

(b) Find the number of lettuces sold by S_2 over the two days.

(a) $M = \begin{bmatrix} 45 & 18 & 11 \\ 35 & 18 & 9 \\ 47 & 29 & 10 \end{bmatrix}$ and $T = \begin{bmatrix} 28 & 13 & 16 \\ 31 & 17 & 13 \\ 29 & 28 & 19 \end{bmatrix}$



On a **Calculator** page, assign *M* and *T* as follows:

- Press [ctrl] [template] to access the **Assign** [:=] command.

*Note: The **Define** command ([menu] > **Actions** > **Define**) and the **Store** command (press [ctrl] [var] to access [sto→]) can also be used.*

- Press [menu] > **Matrix & Vector** > **Create** > **Matrix**.
- Set the number of rows to 3 & the number of columns to 3.
- Enter as shown and calculate $M + T$ as shown.

$$M + T = \begin{bmatrix} 73 & 31 & 27 \\ 66 & 35 & 22 \\ 76 & 57 & 29 \end{bmatrix}$$

***Note**: Alternatively, to create a 3 x 3 matrix, press [catalog] [5], (or [template]) select the **m-by-n Matrix** template and complete as above.*

... continued

Adding and subtracting matrices (continued)

(b) The number of lettuces sold by S_2 over the two days is given by the element $(2,2)$.

To access element $(2,2)$, enter $\boldsymbol{(m+t)[2,2]}$.

Element $(2,2)$ indicates that S_2 sold 35 lettuces on the Monday and the Tuesday.

***Note**: The first element in $[2 \quad 2]$ indicates the row number and the second element indicates the column number.*

Verifying matrix addition properties

The following matrix properties are important where O is the zero matrix:

- $A+B=B+A$
- $A+O=A$
- $A+(-A)=O$

Let $A=\begin{bmatrix} 9 & 4 \\ -2 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 10 & 7 \\ 2 & -4 \end{bmatrix}$.

Verify the following results:

(a) $A+B=B+A$ **(b)** $A+O=A$ **(c)** $A+(-A)=O$

On a **Calculator** page, assign A and B as follows:

- Press ctrl [template key] to access the **Assign** [:=] command.
- Press [book key] 5, select the **2-by-2 Matrix** template and enter as shown.

(a) $A+B=\begin{bmatrix} 19 & 11 \\ 0 & -1 \end{bmatrix}$ and $B+A=\begin{bmatrix} 19 & 11 \\ 0 & -1 \end{bmatrix}$.

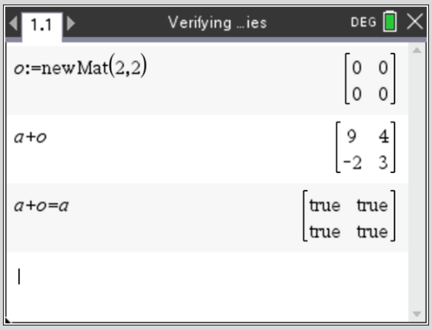
***Note**: Entering $A+B=B+A$ gives the output $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.*

To create a 2 × 2 zero matrix labelled as O:

- Press ctrl [template key] to access the **Assign** [:=] command.
- Press menu > **Matrix & Vector** > **Create** > **Zero Matrix**.
- Enter $\boldsymbol{o := newMat(2,2)}$.

(b) $A+O=\begin{bmatrix} 9 & 4 \\ -2 & 3 \end{bmatrix}=A$.

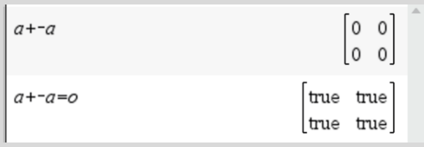
***Note**: Entering $A+O=A$ gives the output $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.*



... continued

Verifying matrix addition properties (continued)

(c) $A+(-A)=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}=O$.



Notes:

1. *Entering* $A+(-A)=O$ *gives the output* $\begin{bmatrix}\text{true} & \text{true}\\\text{true} & \text{true}\end{bmatrix}$.

2. *When attempting to add two matrices of different order, a 'dimension mismatch' error message is displayed.*
An example of this is shown at right. To enter C, *press* [book] [5], *select the* ***2-by-1 Matrix*** *template and enter as shown.*

Defining scalar and matrix multiplication

Matrices can be multiplied by scalar (real number) quantities.

In general, for 2×2 matrices, if $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$ then $kA=\begin{bmatrix}ka & kb\\kc & kd\end{bmatrix}$ where k is a scalar.

Two matrices can be multiplied together if the number of columns in the first matrix equals the number of rows in the second matrix.

In general, if an $m\times n$ matrix is multiplied by an $n\times p$ matrix, the resulting matrix will be of order $m\times p$, that is, $(m\times \boldsymbol{n})\times(\boldsymbol{n}\times p)=m\times p$.

In general, for two matrices, A and B, $AB\neq BA$.

In other words, matrix multiplication in general is not commutative.

The following matrix properties are important:

- $A(B+C)=AB+AC$
- $(B+C)A=BA+CA$

Let $A=\begin{bmatrix}9 & 4\\-2 & 3\end{bmatrix}$, $B=\begin{bmatrix}10 & 7\\2 & -4\end{bmatrix}$, $C=\begin{bmatrix}4 & 0\\-3 & 2\end{bmatrix}$ and $D=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$.

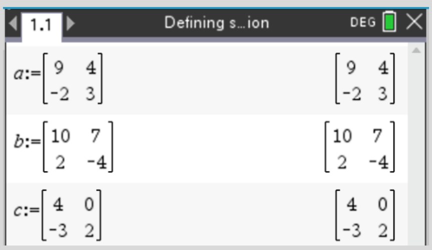
Verify that:

(a) $AB\neq BA$. **(b)** $A(B+C)=AB+AC$. **(c)** $(B+C)A=BA+CA$.

(d) AD can be determined and DA cannot be determined.

On a **Calculator** page, assign A, B and C as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [book] [5], select the **2-by-2 Matrix** template and enter as shown.



(a) $AB=\begin{bmatrix}98 & 47\\-14 & -26\end{bmatrix}$ and $BA=\begin{bmatrix}76 & 61\\26 & -4\end{bmatrix}$.

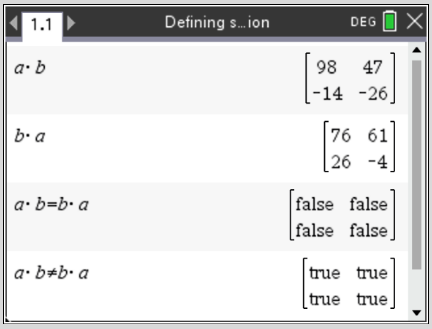
... continued

Defining scalar and matrix multiplication (continued)

***Notes*:**

1. *Entering* $AB = BA$ *gives the output* $\begin{bmatrix} \text{false} & \text{false} \\ \text{false} & \text{false} \end{bmatrix}$ *and entering* $AB \neq BA$ *gives the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.

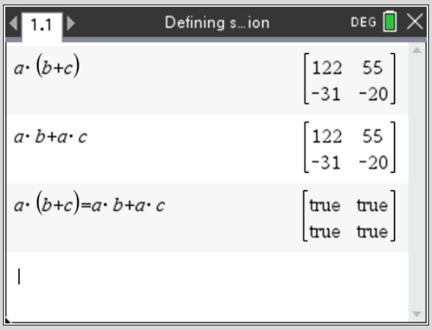
2*. When performing matrix multiplication, always use the multiplication key,* [×]. *Press* [ctrl] [=] *to access the* ≠ *symbol.*



Enter $A(B+C)$ and $AB+AC$ as shown.

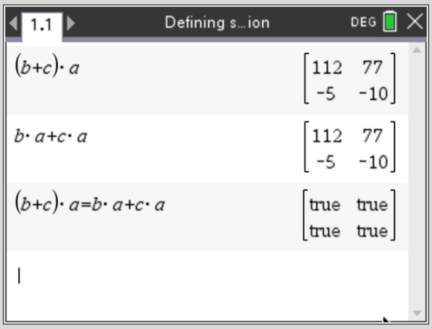
(b) $A(B+C) = \begin{bmatrix} 122 & 55 \\ -31 & -20 \end{bmatrix}$ and $AB + AC = \begin{bmatrix} 122 & 55 \\ -31 & -20 \end{bmatrix}$.

Note: *Entering* $A(B+C) = AB + AC$ *gives the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.



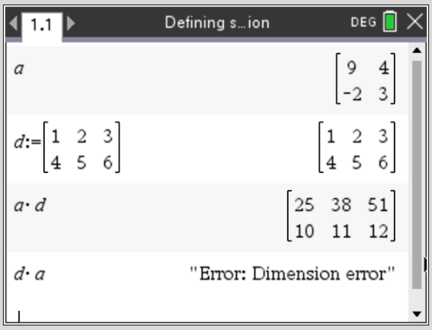
(c) $(B+C)A = \begin{bmatrix} 112 & 77 \\ -5 & -10 \end{bmatrix}$ and $BA + CA = \begin{bmatrix} 112 & 77 \\ -5 & -10 \end{bmatrix}$.

Note: *Entering* $(B+C)A = BA + CA$ *gives the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.



On a **Calculator** page, assign D as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [book] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 2 and the number of columns to be 3.
- Enter as shown.



(d) AD can be determined because A is a 2×2 matrix and D is a 2×3 matrix ($2 = 2$).

However, DA cannot be determined because D is a 2×3 matrix and A is a 2×2 matrix ($3 \neq 2$).

Note*: When attempting to multiply two matrices of different order, a 'dimension error' message is displayed when the number of columns in the first matrix does not equal the number of rows in the second matrix.*

Raising a matrix to a power

Consider $M = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

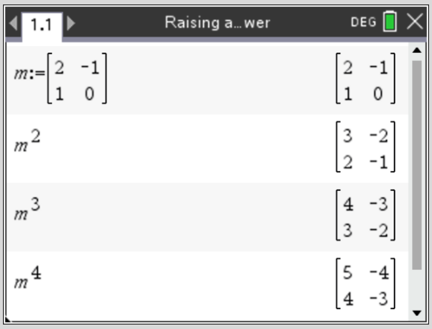
(a) Find M^2, M^3 and M^4.

(b) Hence, infer a general result for M^n where n is a positive integer.

(c) Use your result to determine M^{2025} and check your answer with the *TI-Nspire CX II CAS*.

On a **Calculator** page, assign M as follows:

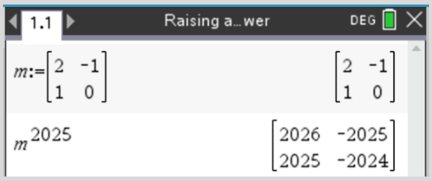
- Press ctrl [templates] to access the **Assign** [:=] command.
- Press [catalog] 5, select the **2-by-2 Matrix** template and enter as shown.



(a) $M^2 = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$, $M^3 = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$ and $M^4 = \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$.

(b) $M^n = \begin{bmatrix} n+1 & -n \\ n & 1-n \end{bmatrix}$.

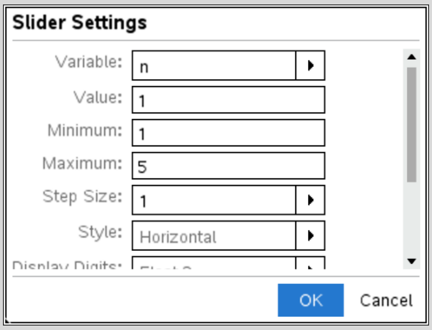
(c) $M^{2025} = \begin{bmatrix} 2025+1 & -2025 \\ 2025 & 1-2025 \end{bmatrix} = \begin{bmatrix} 2026 & -2025 \\ 2025 & -2024 \end{bmatrix}$.



Alternatively, on a **Notes** page:

Insert a **Slider** to control the value of n as follows:

- Press menu > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.



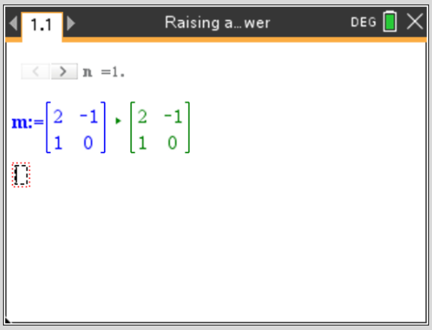
Insert a **Maths Box** as follows:

- Press menu > **Insert** > **Maths Box**.

Note: Alternatively, to insert a ***Maths Box****, press* ctrl M.

Assign M as follows:

- Press ctrl [templates] to access the **Assign** [:=] command.
- Press [catalog] 5, select the **2-by-2 Matrix** template and enter as shown.



... continued

Raising a matrix to a power (continued)

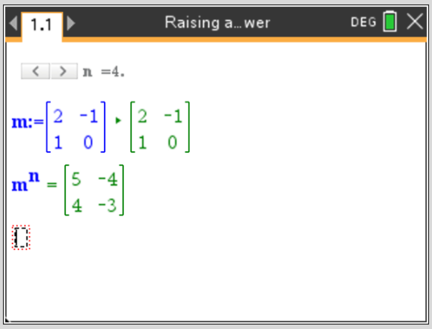
Now:

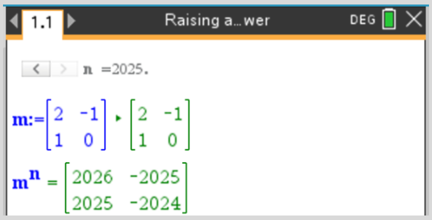
- Insert another **Maths Box** and enter m^n.
- To change the display of a **Maths Box**, for example, to display an equals sign:Click on the **Maths Box**.
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▸ and select $=$.

***Note**: **Maths Box Attributes** can also be accessed within a **Maths Box** by pressing* [ctrl] [menu].

Click on the slider to change the value of n.

To calculate M^n for $n = 2025$, manually change the value of n to 2025 in the slider box.





Using matrix products

Matrix products and powers of matrices can be used to model and solve costing or pricing problems.

A manufacturer makes tables in three sizes: large, medium and small.

The number of units of each component required to make each type of table is shown below.

	labour	*materials*	*machine time*
large	12	10	6
medium	10	8	5
small	9	6	4

The cost of a unit of each component is: labour \$70, materials \$50 and machine time \$40.

(a) Show the unit costs in a 3×1 matrix *C*.

Demand for tables for the next month is as follows: large 10, medium 12 and small 25.

(b) Show the demand in a 1×3 matrix *D*.

(c) Use matrix methods to find, for the next month, the total:

- **(i)** number of units of labour required.
- **(ii)** cost for each type of table.
- **(iii)** cost of materials.

(a) $C = \begin{bmatrix} 70 \\ 50 \\ 40 \end{bmatrix}$ and **(b)** $D = \begin{bmatrix} 10 & 12 & 25 \end{bmatrix}$

... continued

Using matrix products (continued)

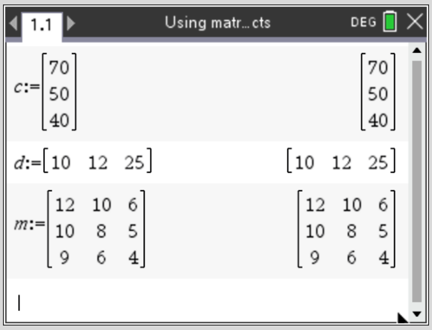
(c) Let $M = \begin{bmatrix} 12 & 10 & 6 \\ 10 & 8 & 5 \\ 9 & 6 & 4 \end{bmatrix}$.

On a **Calculator** page, assign C, D and M as follows:

- Press **ctrl** to access the **Assign** := command.

For matrix C:

- Press **5** and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 1.
- Enter as shown.



For matrix D:

- Press **5** and select the **m-by-n Matrix** template.
- Set the number of rows to be 1 and the number of columns to be 3.
- Enter as shown.

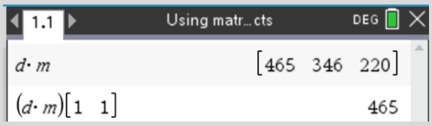
For matrix M:

- Press **5** and select the **m-by-n Matrix** template.
- Set the number of rows to 3 and the number of columns to 3
- Enter as shown.

(c) (i) The total number of units of labour required for the next month is given by element $(1,1)$ in the matrix product DM.

$$DM = \begin{bmatrix} 10 & 12 & 25 \end{bmatrix} \begin{bmatrix} 12 & 10 & 6 \\ 10 & 8 & 5 \\ 9 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 465 & 346 & 220 \end{bmatrix}$$



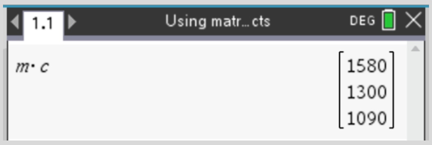
465 units of labour are required for the next month.
To access element $(1,1)$, enter $\boldsymbol{(d \times m)[1,1]}$.

Note: The matrix product DM provides the number of units of labour, materials and machine time required to meet demand.

Note: The first element in $\begin{bmatrix} 1 & 1 \end{bmatrix}$ indicates the row number and the second element indicates the column number. A comma must be entered between the row number and column number, however the comma is hidden in the output.

(c) (ii) The cost for each type of table for the next month is given by the matrix product MC.

$$MC = \begin{bmatrix} 12 & 10 & 6 \\ 10 & 8 & 5 \\ 9 & 6 & 4 \end{bmatrix} \begin{bmatrix} 70 \\ 50 \\ 40 \end{bmatrix} = \begin{bmatrix} 1580 \\ 1300 \\ 1090 \end{bmatrix}$$



The cost for each type of table is: large \$1580, medium \$1300 and small \$1090.

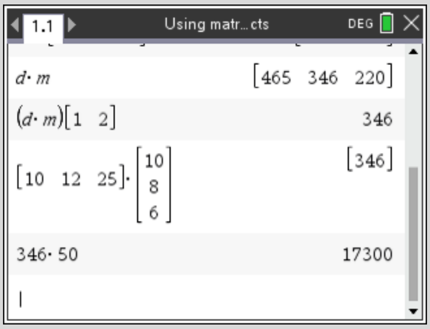
... continued

Using matrix products (continued)

(c) (iii) The total cost of materials for the next month is given by $50 \times$ element $(1,2)$ in the matrix product DM.

To access element $(1,2)$, enter $(\boldsymbol{d \times m})[\mathbf{1,2}]$.

$[10 \quad 12 \quad 25]\begin{bmatrix}10\\8\\6\end{bmatrix} = [346]$ and so 346 units of materials are required. The cost of a unit is \$50, so the total cost is \$17 300.



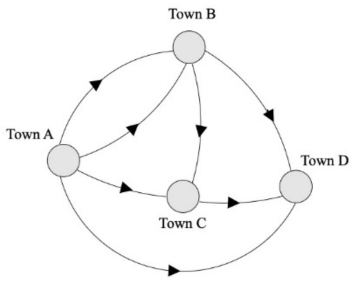
Using matrix powers

Squaring a matrix can be used to determine the number of ways pairs of people in a network can communicate with each other via a third person.

A network is a finite set of objects (represented by dots) called vertices with connecting links called edges. Examples of networks include towns which may be connected by roads.

Derek delivers pizzas to four towns, A, B, C and D.

The network diagram below shows the one-way road connections between A, B, C and D.



(a) Represent the one-stage pathways that Derek could take as a 4×4 matrix A.

(b) Represent the two-stage pathways that Derek could take as a 4×4 matrix A^2.

(a) There are 2 one-stage pathways from A to B, one from A to C, one from A to D, one from B to C, one from B to D and one from C to D.

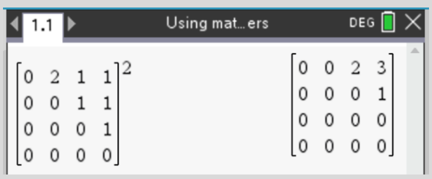
$$A = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(b) There are 2 two-stage pathways from A to B to C, 3 two-stage pathways from A to D (2 are A to B to D and the other is A to C to D) and one from B to C to D.

$$A^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

To confirm this on a **Calculator** page:

- Press [📖] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 4 and the number of columns to be 4.
- Enter as shown.



Verifying identity and inverse matrix properties

The following matrix properties are important where I is the identity matrix.

- $AI = A = IA$
- $AA^{-1} = I = A^{-1}A$

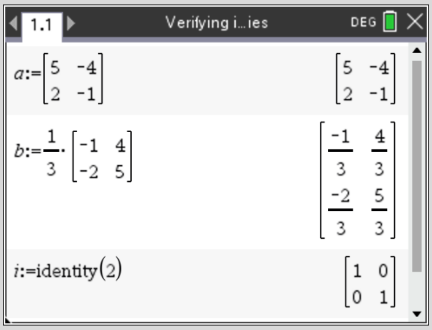
Let $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$ and $B = \frac{1}{3}\begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$.

Verify the following results:

(a) $AI = A = IA$ **(b)** $AB = I = BA$ hence show $B = A^{-1}$

On a **Calculator** page, assign A and B as follows:

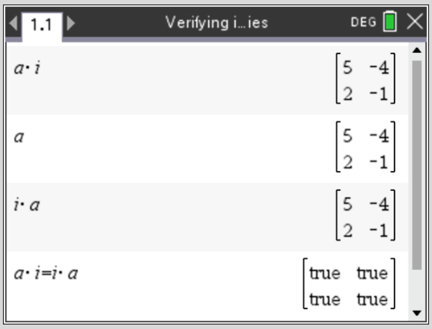
- Press ctrl ⊞ to access the **Assign** := command.
- Press 📖 5, select the **2-by-2 Matrix** template and enter as shown.



To create I where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

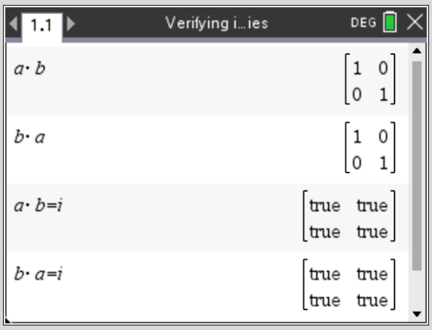
- Press ctrl ⊞ to access the **Assign** := command.
- Press menu > **Matrix & Vector** > **Create** > **Identity**.
- Enter as shown.

***Note**: For the following verification questions, the **Calculation Mode** must be set to either **Auto** or **Exact**. This can be done via* ⌂on *> **Settings** > **Document Settings**.*



(a) $AI = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$, $IA = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$ and $AI = A = IA$.

***Note**: Entering* $AI = IA$ *gives the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.



(b) $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

***Note**: Entering* $AB = I$ *and* $BA = I$ *both give the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.

From part (b), it can be concluded that $B = A^{-1}$.

Calculating the determinant and inverse of 2 x 2 matrices

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A)=ad-bc$.

$$A^{-1}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ where } \det(A)\neq 0\ (ad-bc\neq 0).$$

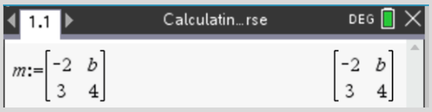
If $\det(A)=0$, then A is a singular matrix and A^{-1} does not exist.

Consider $M=\begin{bmatrix} -2 & b \\ 3 & 4 \end{bmatrix}$ where $\det(M)=-14$.

(a) Find the value of b. **(b)** Find M^{-1}.

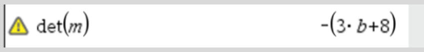
On a **Calculator** page, assign M as follows:

- Press ctrl [template key] to access the **Assign** [:=] command.
- Press [book key] 5, select the **2-by-2 Matrix** template and enter as shown.



To find $\det(M)$ in terms of b:

- Press menu > **Matrix & Vector** > **Determinant** and enter as shown.

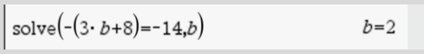


(a) $\det(M)=-(3b+8)$

Solve the linear equation $-(3b+8)=-14$ for b as follows:

- Press menu > **Algebra** > **Solve**.
- Enter as shown.

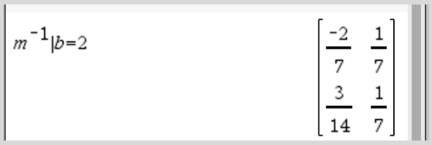
Solving $-(3b-8)=-14$ for b gives $b=2$.



***Note**: The command **solve(det(m) = –14,b)** could also be used here.*

To calculate M^{-1} with $b = 2$, enter as shown.

- Press ctrl = to access the 'with' or 'given' symbol |.



(b) $M^{-1}=-\frac{1}{14}\begin{bmatrix} 4 & -2 \\ -3 & -2 \end{bmatrix}=\begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}$.

Notes:

***1.** To find M^{-1} without using TI-Nspire CX II CAS, interchange the elements on the leading (main) diagonal, change the signs of the elements on the secondary diagonal and divide each element by $\det(M)=-14$.*

***2.** To check your answer, calculate MM^{-1} or $M^{-1}M$. M and M^{-1} should satisfy $MM^{-1}=M^{-1}M=I$.*

1.4.3. Solving with matrices

Solving matrix equations involving matrices of up to dimension 2 x 2

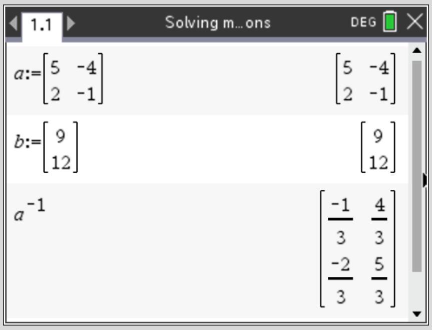
Inverses can be used to solve matrix equations. If $AX = B$, where A is a square matrix and has inverse A^{-1} such that $A^{-1}A = I$, then the solution is $X = A^{-1}B$.

If $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$ and X is a matrix such that $AX = B$, find:

(a) A^{-1}. **(b)** X.

On a **Calculator** page, assign A and B as follows:

- Press ctrl [templates] to access the **Assign** [:=] command.
- Press [catalog] 5, select the **2-by-2 Matrix** template for A and enter as shown.
- Press [catalog] 5, select the **2-by-1 Matrix** template for B and enter as shown.

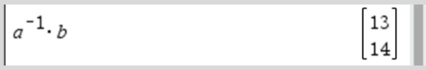


To calculate A^{-1} enter as shown.

(a) $A^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$.

(b) Since $AX = B$, then $X = A^{-1}B$.

$X = \frac{1}{3}\begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \end{bmatrix}$



Solving systems of linear equations with matrices

Use the **Simultaneous** command to solve the following system of linear equations

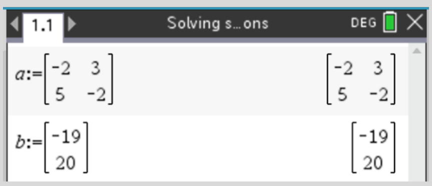
$$\begin{aligned} -2x + 3y &= -19 \\ 5x - 2y &= 20 \end{aligned}.$$

In matrix form , the system of equations can be expressed as

$AX = B$ where $A = \begin{bmatrix} -2 & 3 \\ 5 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -19 \\ 20 \end{bmatrix}$.

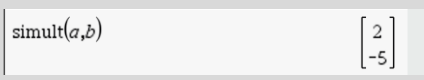
On a **Calculator** page, assign A and B as follows:

- Press ctrl [templates] to access the **Assign** [:=] command.
- Press [catalog] 5, select the **2-by-2 Matrix** template for A and enter as shown.
- Press [catalog] 5, select the **2-by-1 Matrix** template for B and enter as shown.



To solve this system of linear equations using the **Simultaneous** command:

- Press menu > **Matrix & Vector** > **Simultaneous** and enter as shown.



So $x = 2$ and $y = -5$.

Applying inverse matrices

A manufacturer makes tables in three sizes: large, medium and small.

The number of units of each component required to make each type of table is shown below.

	labour	*materials*	*machine time*
large	12	10	6
medium	10	8	5
small	9	6	4

The manufacturer is concerned that tax rises and overheads are eating into the company's profits.
Originally, the cost for each type of table was large \$1580, medium \$1300 and small \$1090.
It is decided to increase the cost of each table to: large \$1810, medium \$1490 and small \$1250.
The same number of units of each component required to make each type of table is unchanged.

Use matrix methods to find the new unit costs that are being used.

Let $M = \begin{bmatrix} 12 & 10 & 6 \\ 10 & 8 & 5 \\ 9 & 6 & 4 \end{bmatrix}$ and $T = \begin{bmatrix} 1810 \\ 1490 \\ 1250 \end{bmatrix}$.

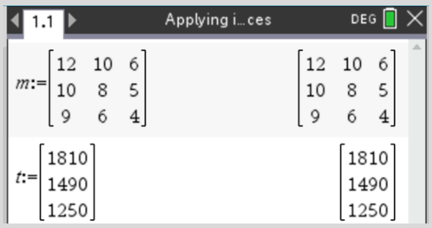
On a **Calculator** page, assign M and T as follows:

For matrix M:

- Press [book] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 3.
- Enter as shown.

For matrix T:

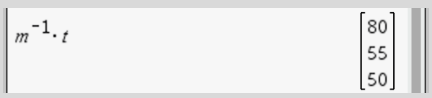
- Press [book] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 1.
- Enter as shown.



$MX = T$ and so $X = M^{-1}T$

$$\begin{bmatrix} 12 & 10 & 6 \\ 10 & 8 & 5 \\ 9 & 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1810 \\ 1490 \\ 1250 \end{bmatrix} \text{ and so}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 & 10 & 6 \\ 10 & 8 & 5 \\ 9 & 6 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 1810 \\ 1490 \\ 1250 \end{bmatrix} = \begin{bmatrix} 80 \\ 55 \\ 50 \end{bmatrix}.$$



The new costs are: labour \$80, materials \$55 and machine time \$50.

1.4.4. Transition matrices

Introducing transition matrices

Transition matrices are used to predict what will happen in the future based on what has happened previously and the likelihood of an event occurring given what has happened previously.

Transition matrices are useful when we wish to model situations where:

- the conditions or states are clearly defined sets.
- there is a transition from one state to the next.
- the next state depends only on the previous one.

The following shows a typical table used to form a transition matrix.

	Current states	
Next states	Event A	Event B
Event A	0.6	0.27
Event B	0.4	0.73

The associated transition matrix is:

$$T = \begin{bmatrix} 0.6 & 0.27 \\ 0.4 & 0.73 \end{bmatrix}$$

The sum of each column is 1.

The initial state matrix S_0 gives the numbers or proportions of the objects initially in each state.

The transition matrix multiplies the state matrix to form a new state matrix.

The state matrix formed after applying the transition matrix once is S_1, after it has been applied twice is S_2 and so on.

The state matrix after n transitions can be found using $S_n = T^n S_0$.

If the previous state is known, the next state can be found using $S_{n+1} = TS_n$.

As the number of transitions increases indefinitely, the proportions in each category will approach a steady state.

In other words, as n gets large, S_n stabilises to a steady state.

The following example showcases an algebraic approach to finding steady state values.

No change in successive state matrices is strongly indicative that steady state has been reached.

It is a good idea to always check by finding a state matrix far into the future.

There are two airline companies flying between *Letsgo* and *Arewethereyet*: *Company A* and *Company B*.

Passengers are surveyed after a flight and it is found that 40% of *Company A*'s passengers indicate they will fly with *Company A* next time they fly and 30% of *Company B*'s passengers indicate that they will fly with *Company B* next time they fly.

... continued

Introducing transition matrices (continued)

On the day the survey is taken, *Company A* has 300 passengers and *Company B* has 500 passengers.

(a) Construct the transition matrix T.

(b) Write down the initial state matrix S_0.

(c) Find **(i)** S_1. **(ii)** S_2. **(iii)** S_3. **(iv)** S_4.

(d) What values do the number of passengers with each airline seem to be approaching?

If A, the number who fly with *Company A*, and B, the number who fly with *Company B*, have reached steady state values, multiplying by the transition matrix will not change them.

To calculate these steady state values, consider the matrix equation: $\begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$

Expanding the matrices and simplifying the two resulting equations, it is found that both equations simplify to give the same equation.

(e) Find the single equation that results when each of the above equations is simplified.

(f) Hence state the value of A in terms of B.

(g) What proportion of passengers will *Company A* expect to have in the long term?

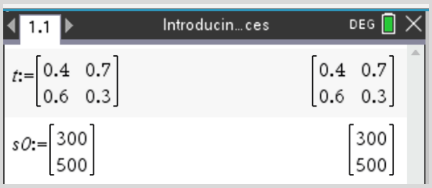
(h) Of the original 800 passengers, how many will each company expect to have in the long term?

(a) The transition matrix is given by $T = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}$.

(b) The initial state matrix is given by $S_0 = \begin{bmatrix} 300 \\ 500 \end{bmatrix}$.

On a **Calculator** page, assign T and S_0 as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **2-by-2 Matrix** template for T and enter as shown.
- Press [catalog] [5], select the **2-by-1 Matrix** template for S_0 and enter as shown.



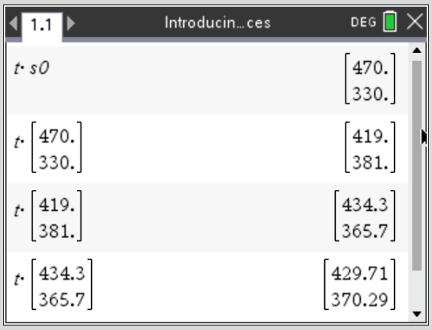
(c) On a **Calculator** page (answers given to nearest integers):

(i) $S_1 = TS_0 = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}\begin{bmatrix} 300 \\ 500 \end{bmatrix} = \begin{bmatrix} 470 \\ 370 \end{bmatrix}$

(ii) $S_2 = TS_1 = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}\begin{bmatrix} 470 \\ 370 \end{bmatrix} = \begin{bmatrix} 419 \\ 381 \end{bmatrix}$

(iii) $S_3 = TS_2 = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}\begin{bmatrix} 419 \\ 381 \end{bmatrix} = \begin{bmatrix} 434.3 \\ 365.7 \end{bmatrix} = \begin{bmatrix} 434 \\ 366 \end{bmatrix}$.

(iv) $S_4 = TS_3 = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}\begin{bmatrix} 434.3 \\ 365.7 \end{bmatrix} = \begin{bmatrix} 429.71 \\ 370.29 \end{bmatrix} = \begin{bmatrix} 430 \\ 370 \end{bmatrix}$.



***Note**: Alternatively, powers of the transition matrix can be used. See after part **(h)** for an alternative approach based on the powers of the transition matrix using a **Notes** page.*

... continued

Introducing transition matrices (continued)

(d) For *Company A*, the numbers seem to be approaching 430, and for *Company B*, approximately 370.

(e) Given $\begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$.

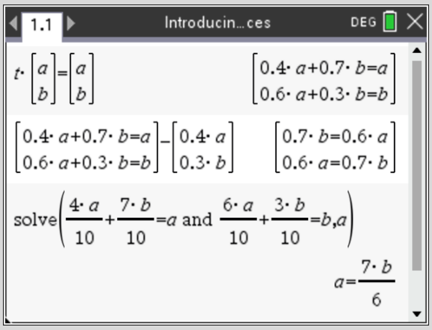
Expanding the two equations gives:

$0.4A + 0.7B = A$ (1)

$0.6A + 0.3B = B$ (2)

Subtracting $0.4A$ from both sides in (1) gives $0.7B = 0.6A$.

Subtracting $0.3B$ from both sides in (2) gives $0.7B = 0.6A$.



(f) Determine A in terms of B.

- Press menu > **Algebra** > **Solve** and enter as shown.

So $A = \frac{7}{6}B$.

(g) The number of passengers flying with each company in the long term is in the ratio 7:6.

So $\frac{7}{13}$ of passengers are expected to fly with *Company A* (and $\frac{6}{13}$ with *Company B*).

(h) *Company A*: $\frac{7}{13} \times 800 = 431$, to the nearest integer.

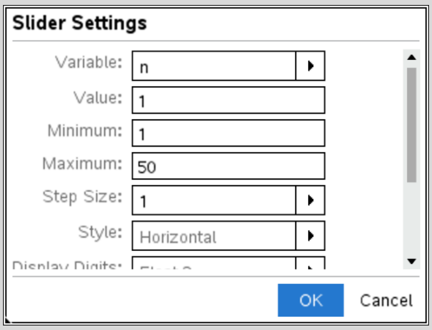
Company B: $\frac{6}{13} \times 800 = 369$, to the nearest integer.

Alternatively, $800 - 431 = 369$.

An alternative approach to parts **(c)** and **(h)** follows:

On a **Notes** page:

Insert a **Slider** to control the value of n as follows:



- Press menu > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.

Insert a **Maths Box** as follows:

- Press menu > **Insert** > **Maths Box**.

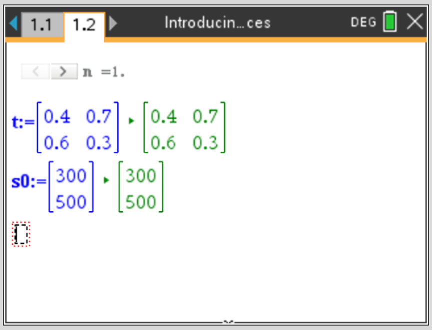
Note: Alternatively, to insert a ***Maths Box****, press* ctrl M.

... continued

Introducing transition matrices (continued)

Assign T as follows:

- Press ctrl to access the **Assign** $:=$ command.
- Press 5, select the **2-by-2 Matrix** template and enter as shown.



Assign S_0 as follows:

- Press ctrl to access the **Assign** $:=$ command.
- Press 5, select the **2-by-1 Matrix** template and enter as shown.

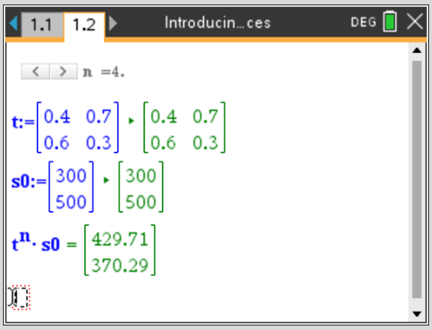
Assign S_n as $T^n \cdot S_0$:

- Press ctrl M to insert a **Maths Box**
- Press ctrl to access the **Assign** $:=$ command and enter as shown.

To change the display of a **Maths Box**, for example, to display an equals sign:

- Click on the **Maths Box**.
- Press menu > **Maths Box Options** > **Maths Box Attributes**.
- Press tab to highlight the **Insert Symbol** field.
- Press ▶ and select $=$.

***Note: Maths Box Attributes** can also be accessed within a **Maths Box** by pressing ctrl menu.*



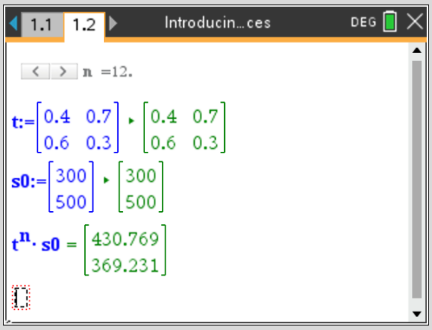
(c) Click on the slider to change the value of n and hence determine S_1, S_2, S_3 and S_4.

(i) $S_1 = TS_0 = \begin{bmatrix} 470 \\ 370 \end{bmatrix}$ **(ii)** $S_2 = T^2 S_0 = \begin{bmatrix} 419 \\ 381 \end{bmatrix}$

(iii) $S_3 = T^3 S_0 = \begin{bmatrix} 434 \\ 366 \end{bmatrix}$, to the nearest integers.

(iv) $S_4 = T^4 S_0 = \begin{bmatrix} 430 \\ 370 \end{bmatrix}$, to the nearest integers.

(h) While no change in successive state matrices is strongly indicative that steady state has been reached, we should always check by finding a state matrix far into the future.



Click on the slider to change the value of n and observe the behaviour of successive state matrices.

Correct to three decimal places, the state matrix elements first stop changing from S_{11} to S_{12}.

Company A: 431, to the nearest integer.

Company B: 369, to the nearest integer.

VCE General Mathematics Unit 2

2.1. Investigating relationships between two numerical variables

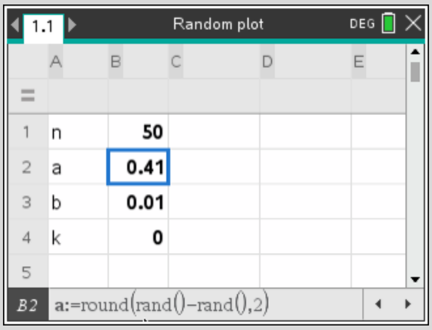
2.1.1. Analysing the association between two numeric variables

Generating random scatter plots

It can be useful to generate scatter plots when analysing association between two numeric variables. In the following example, a scatter plot is generated with the help of the random number generator, and a slider for controlling the 'extent' of this randomness to be applied.

To generate a random set of points on a **Lists & Spreadsheet** page:

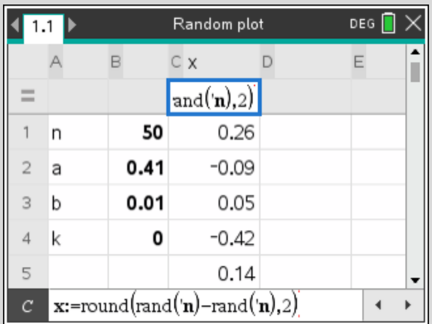
- In column A, enter the letters *n*, *a*, *b*, and *k* as shown.
- In column B:
  - For *n*, enter ***n*:=50**.
  - For *a*, enter ***a*:=round(rand()–rand(),2)**.
  - For *b*, enter ***b*:=round(rand()–rand(),2)**.
  - For *k*, enter ***k*:=0**.



Note: *The* ***rand()*** *function generates a random number between 0 and 1, so the command* ***rand()–rand()*** *will generate a number between –1 and 1.*

To generate *n* random values for *x*, in column C:

- In the column C heading cell, enter the name ***x***.
- In the column C formula cell, type the formula **=round(rand('n)-rand('n),2)**

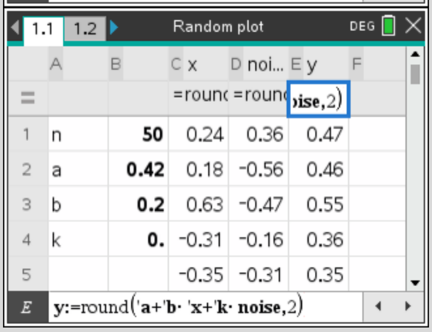


To generate values for *noise*, in column D:

- In the column D heading cell, enter the name ***noise***.
- In the column D formula cell, type the formula **=round(rand('n)-rand('n),2)**

To generate values for *y*, in column E:

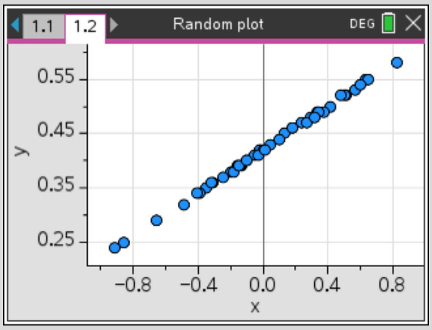
- In the column E heading cell, enter the name ***y***.
- In the column E formula cell, type the formula **=round('a+'b×'x+'k×noise,2)**



Note: *The single 'dash' character in the above formulae can be found via the* [?!▸] *key. It is used here to indicate a variable reference rather than a column reference.*

Add a **Data & Statistics** page, and then:

- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select the variable ***x***.
- Press [tab] to activate **Click to add variable** to the left of the vertical axis and select the variable ***y***.

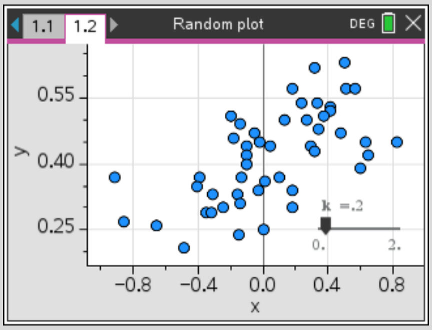


This will display a scatter plot with a very linear appearance, since $k = 0$ and any 'imperfections' will be related to the rounding of values.

... continued

To add a slider on the plot controlling the value of k:

- Press [menu] > **Actions** > **Add Slider**.
  In the dialog box that follows, enter the values:
  Variable: ***b*** Value: **0** Minimum: **0**
  Maximum: **2** Step Size: **0.1**
- Press [enter] to display the slider and position it as required.



Click on the slider and use the arrow keys to change the value of k to vary the strength of association, and press [menu] > **Window/Zoom** > **Zoom-Data** to redraw the window.

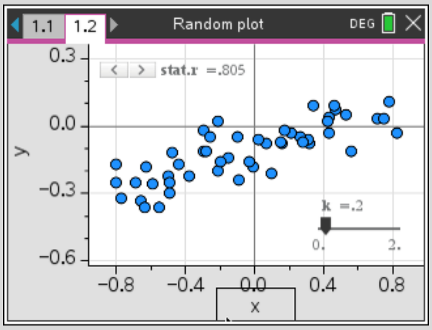
To generate a new plot:

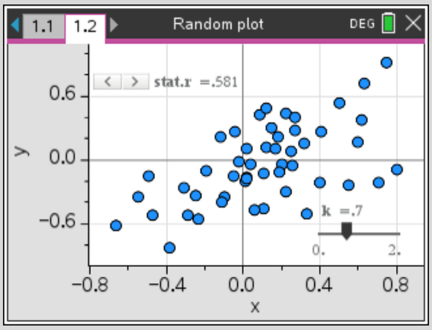
- Press [ctrl] ◀ to display the **Lists & Spreadsheet** page.
- Press [ctrl] [R] to recalculate all the formulae and generate another random plot.
- Press [ctrl] ▶ to display the new plot on the **Data & Statistics** page.
- Press [menu] > **Window/Zoom** > **Zoom-Data** to redraw the window if needed.

Extension

It is also possible to display the value of the correlation coefficient for each scatter plot. To do this:

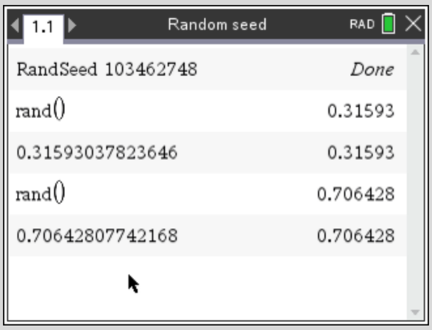
- Press [ctrl] ◀ to display the **Lists & Spreadsheet** page.
- Press [menu] > **Statistics** > **Stat Calculations** > **Two-Variable Staistics**
  - For X List, click to select '*x*.
  - For Y List, click '*y*.
  - Press [enter] to calculate and display the summary statistics for these two variables in the spreadsheet.
- Press [ctrl] ▶ to display the **Data & Statistics** page.
- Press [menu] > **Actions** > **Add Slider**.
  In the dialog box that follows, select the variable: ***stat.r*** and click the minimised option.
- Press [enter] to display the slider and position it as required.





If a 'recalculation' is done, or the value of k is changed, the new value of the correlation coefficient r is displayed. It is **not** intended that you interact with the slider directly.

Note: *The 'random' number generator used on the TI-Nspire CX II CAS (and most mathematical software) is a 'pseudo-random' generator, because it simulates a sequence of random numbers from a known sequence. One way to highlight this is to show how a calculator that has been reset will always give the same starting value as an output for its* ***rand*** *function. For this reason, it is best to 'seed' the random number function so that it starts from a different spot in the sequence. An example of seeding is shown in the screen shown right. The* ***RandSeed*** *command can be found by pressing* [menu] ***> Probability > Random > Seed****.*



2.1.2. Lines of good fit

Calculating the slope and y-intercept of a line of best fit given two points

A given scatter plot shows that the line of best fit will approximately pass through the points (2,3) and (6,11). Find the slope and *y*-intercept of such a line.

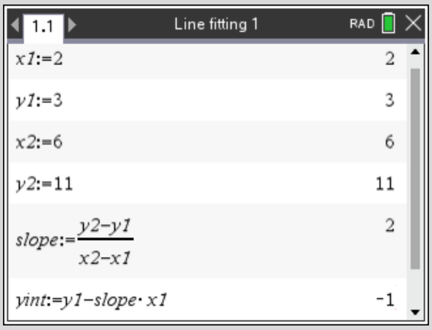
On a **Calculator** page:

- Enter the values for **(*x*1, *y*1)** and **(*x*2, *y*2)** as shown.

To calculate the slope and *y*-intercept of the line passing through both points:

- Enter $slope := \dfrac{y2 - y1}{x2 - x1}$.
- Enter $yint := y1 - slope \times x1$.

Answer: The slope is 2 and the *y*-intercept is –1.



Constructing a notes template to find the equation of the line of best fit given two points

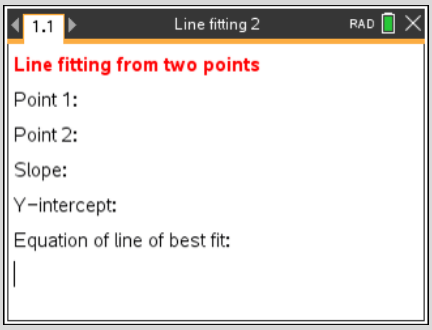
A **Notes** page can be constructed to calculate the equation of the line of best fit for a scatter plot (using the form $y = a + bx$) given two suitable points on a plot. This will be tested with the sample points (2,3) and (6,11).

On a **Notes** page:

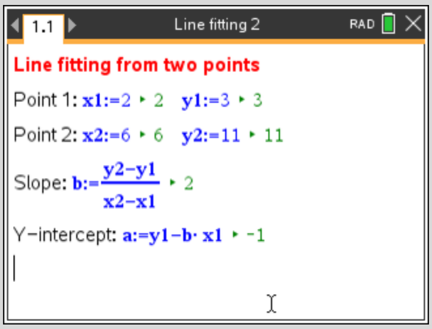
- Enter the text shown in the screenshot.
- Move the cursor to the right of the word ‘Point 1:’ and press [menu] **> Insert > Maths Box** (or press [ctrl] [M]).

Repeat to insert **Maths Boxes** next to each of the other template headings.

Note: *To edit the text colour, select the text by holding* [⇧shift] *and ‘arrow’ across the text. Then press* [menu] ***> Format > Text colour***.

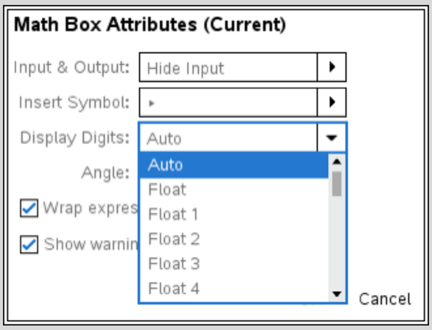


- Click on the **Maths Box** next to the word ‘Point 1’.
- Inside the **Maths Box**, input $x1 := 2$
- Repeat this method to enter the following:
  - For the *y*-coordinate of Point 1, enter $y1 := 3$
  - For the *x*-coordinate of Point 2, enter $x2 := 6$
  - For the *y*-coordinate of Point 2, enter $y2 := 11$
  - For ‘Slope:’, enter $b := \dfrac{y2 - y1}{x2 - x1}$.
  - For ‘*Y*-intercept:’, enter $a := y1 - b \times x1$.



Answer: The equation of the line passing through the points with coordinates (2,3) and (6,11) is $y = -1 + 2x$.

Note: *Unless otherwise specified, the output values will be displayed using the system settings for ‘Display Digits’. However, the display precision of each Maths Box can be set individually using* [menu] ***> Maths Box Options > Math Box Attributes.***



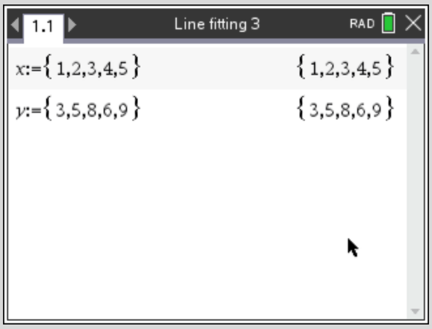
Finding the equation of the line of best fit by minimising the sum of the areas of squares

The least squares regression method can be introduced to students by visualising the process of minimising the areas of the residual squares, using the approach outlined below.

To enter sample points for a scatter plot on a **Calculator** page:

- Enter ***x*:= {1,2,3,4,5}**.
- Enter ***y*:= {3,5,6,8,9}**.

Note: *To enter the ‘Assign’ symbol (i.e. ‘:=”), press* **ctrl**.

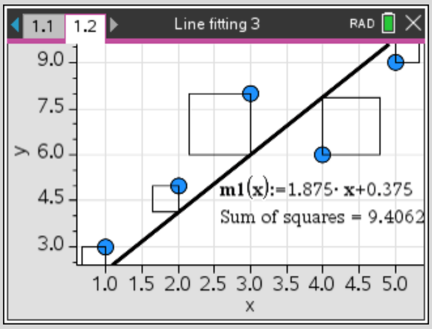


Add a **Data & Statistics** page, and then:

- Press **tab** to activate **Click to add variable** underneath the horizontal axis and select the variable ***x***.
- Press **tab** to activate **Click to add variable** to the left of the vertical axis and select the variable ***y***.

This will display a scatter plot. To add a ‘movable’ line, and display residual squares:

- Press **menu** > **Analyse** > **Add Movable Line**.
- Press **menu** > **Analyse** > **Residuals** > **Show Residual Squares**.



This will display a line and its equation nearby the plotted points. It also displays the squares formed by the vertical distance between each point and the line, as well as the total area of these squares (the ‘Sum of squares’).

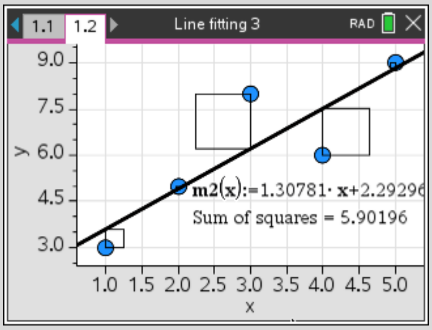
The line can be ‘moved’ in two possible ways.

To change the *slope* of the line:

- Hover the cursor near to the ‘ends’ of the line, then click and drag to change the *slope* of the line and note how the equation and total area of the squares is changed.

To change the *y-intercept* of the line:

- Hover the cursor near to the ‘middle’ of the line, then click and drag to change the *y-intercept* of the line and note how the equation and total area of the squares is changed.



By making appropriate ‘moves’, it is possible to reduce the ‘Sum of squares’ to below 6 square units. For reference, the least squares algorithm gives the equation of the line as $y = 1.3x + 2.3$, with the minimum ‘sum of squares’ as 5.9.

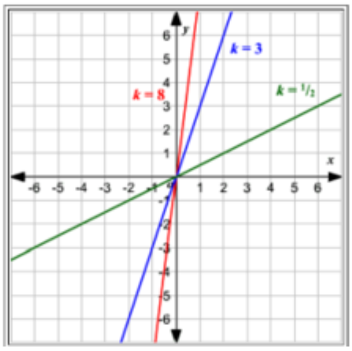
Note: *To stop using the ‘movable line’, press* **menu** > ***Analyse > Remove Movable Line.***

2.2. Variation

2.2.1. Direct Variation

Finding the constant of proportionality

Direct variation of the form $y \propto x$ implies $y = kx,\ k > 0$. Graphically, this type of variation is characterised by a straight-line graph passing through the origin $(0, 0)$. The diagram on the right illustrates three examples of direct variation graphs all with the form $y = kx, k > 0$.



The value of $k = \dfrac{y}{x}$ is referred to as the constant of proportionality.

Consider the following example. At the local farmer's market, you saw someone purchase 2 kilograms of strawberries and pay \$39.20. You want to buy strawberries too, but you want only 0.75 kilograms. How much would you expect to pay?

To determine the value of *k*, the constant of proportionality, on a **Calculator** page:

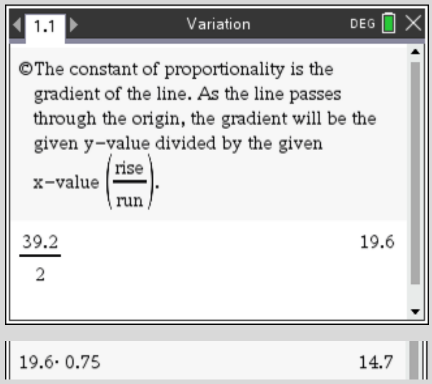
- Enter $\mathbf{\dfrac{39.2}{2}}$

Answer*:* $k = 19.6$.

- Press **19.6 × 0.75** to calculate the cost of 0.75 kg of strawberries.

Answer: The cost is $\$14.70$.

Note*: Using* ***Ans × 0.75*** *or will produce the same result. Also, if only '****× 0.75'*** *is entered, the calculator assumes that the previous result is to be multiplied by 0.75.*



Completing a table of values for the response variable

Find the missing values given $d \propto t$.

t	4	7.3	26
d		15.33	

We know that if $d \propto t$ then $d = k \times t$.

To find the missing values we first need to find the constant of proportionality, k, on a **Calculator** page as above:

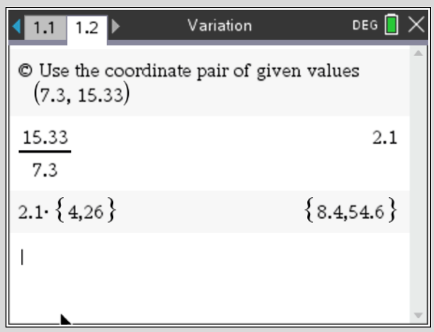
- Enter the fraction $\mathbf{\dfrac{15.33}{7.3}}$.

Answer*:* $k = 2.1$.

To find a set of values for the response variable *d*, we can use the braces (set brackets) via ctrl) as follows:

- Enter **2.1 × {4,26}**

Answer*:* The missing values are $\{8.4, 54.6\}$ respectively.



Determining a value for the explanatory variable

Find the missing value given $s \propto p$.

p	12	
s	31.2	96.2

To find the missing values we first need to find the constant of proportionality, k, on a **Calculator** page as above:

- Enter $\mathbf{\frac{31.2}{12}}$.

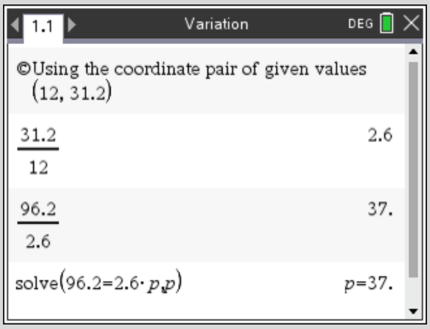
Answer: $k = 2.6$.

We can now write the equation $s = 2.6p$.
To find p when s = 96.2:

- Enter $\mathbf{\frac{96.2}{2.6}}$ (or **96.2 ÷ Ans**).

Alternatively, enter **solve(96.2 = 2.6*p*,*p*).**

Answer: $p = 37$.



2.2.2. Inverse Variation

Finding the constant of proportionality

Inverse variation is when the response variable decreases as the explanatory variable increases such that $y \propto \frac{1}{x} \Rightarrow y = \frac{k}{x}$. Rearranging this we see that $x \times y = k$ which means that the product of the coordinate pair will always be constant.

Consider the following example. A team of two plasterers are working on a house. They will take five days to complete the job. How long would it take a team of three plasterers to complete the same job?

plasterers	2	3
days	5	

To determine the constant of proportionality *k*, on a **Calculator** page:

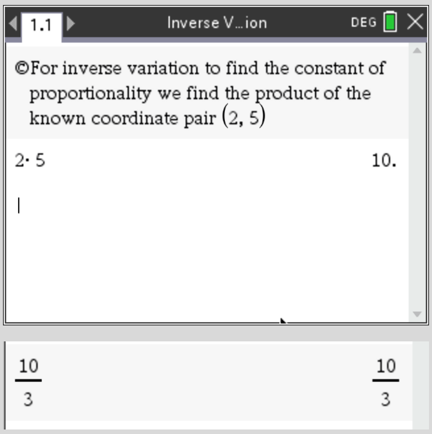
- Enter **2 × 5**.

Answer: $k = 10$, i.e. $days = \frac{10}{plasterers}$.

To calculate the number of days it would it take a team of three plasterers to complete the same job.

- Enter $\mathbf{\frac{10}{3}}$(or **Ans ÷ 3**)

Answer: It would it take a team of three plasterers $\frac{10}{3} = 3.\dot{3}$ days to complete the same job.



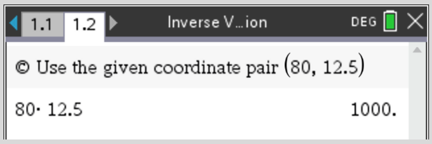
Completing a table of values for the response variable

Find the missing values in the table below, given $time \propto \frac{1}{speed}$.

speed (km/hr)	70	80	110
time (hrs)		12.5	

To determine the constant of proportionality *k*, on a **Calculator** page:

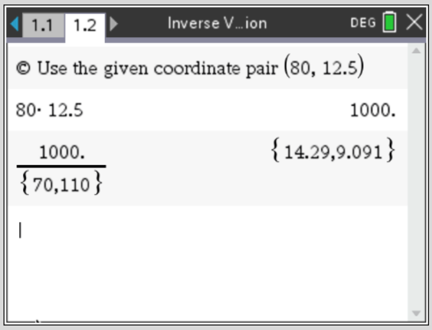
- Enter **80 × 12.5**.



Answer: $k = 1000$, so $time = \frac{k}{speed} = \frac{1000}{speed}$.

To find a set of values for the response variable *time*, we can use the braces (set brackets) via ctrl) as follows:

- Enter $\mathbf{\frac{1000}{\{70,110\}}}$



Answer: The missing values are $\{14.29, 9.091\}$ respectively.

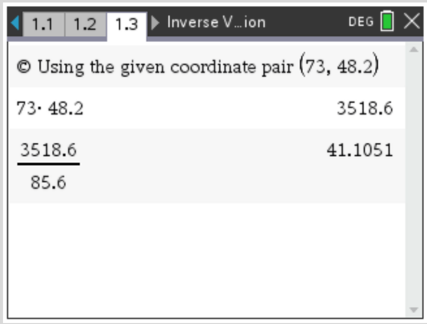
Determining a value for the explanatory variable

Find the missing value given $r \propto \frac{1}{m}$. Recall that $r = \frac{k}{m} \Rightarrow k = m \times r$.

m	73	
r	48.2	85.6

To determine the constant, *k* on a **Calculator** page:

- Enter **73 × 48.2**.



Answer: $k = 3518.6$, i.e. $r = \frac{3518.6}{m}$ but also $m = \frac{3518.6}{r}$.

To calculate the missing value:

- Enter $\mathbf{\frac{3518.6}{85.6}}$ (or **Ans ÷ 85.6**)

Answer: $m = 41.11$ to 2 decimal places.

2.2.3. Transformations and non-linear models

Linearising scatter plots

There are three transformations of the explanatory variable (x) which may 'linearise' a scatter plot (make the scatter plot appear more line-like), and provide support for a potential non-linear model for a relationship between two numeric variables:

- **Squared x transformation**: If $y \propto x^2 \Rightarrow y = kx^2 \Leftrightarrow k = \dfrac{y}{x^2}$, so plotting y against x^2 should linearise the scatter plot.
- **Logarithm of x transformation:** If $y \propto \log_{10}(x) \Rightarrow y = k\log_{10}(x) \Leftrightarrow k = \dfrac{y}{\log_{10}(x)}$, so plotting y against $\log_{10}(x)$ should linearise the scatter plot.
- **Reciprocal of x transformation**: If $y \propto \dfrac{1}{x} \Rightarrow y = \dfrac{k}{x} \Leftrightarrow k = xy$, so plotting y against $\dfrac{1}{x}$ should linearise the scatter plot.

If we plot the y values against the transformed x values, a straight line through the origin can be fitted with gradient k.

***Note**: For the following questions, it will be assumed that the **Calculation Mode** is set to **Approximate**. This can be done via [on] > **Settings** > **Document Settings**.*

(a) If $y \propto x^2$, use a **Calculator** page to complete the table and hence find an equation that links y and x. Verify this equation by using a least squares regression line over a scatter plot of y vs. x^2.

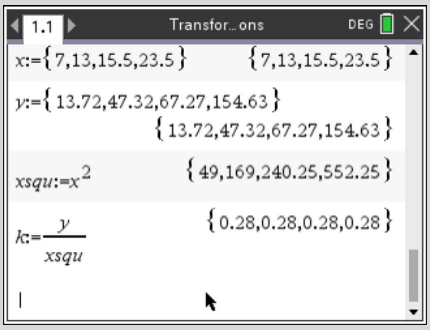
x	7	13	15.5	23.5
y	13.72	47.32	67.27	154.63
x^2				
$k = \dfrac{y}{x^2}$				

(b) If $y \propto \log_{10}(x)$, use a **Lists & Spreadsheet** page to complete the table and hence find an equation that links y and x. Verify this equation by using a least squares regression line over a scatter plot of y vs. $\log_{10}(x)$.

x	14	28	70	140
y	4.126	5.210	6.640	7.726
$\log_{10}(x)$				
$k = \dfrac{y}{\log_{10}(x)}$				

(a) On a **Calculator** page, assign x and y as follows:

- Press [ctrl] [template] to access the **Assign** [:=] command.
- For x, enter ***x*:={7,13, 15.2, 23.5}**.
- For y, enter ***y*:={13.72,47.32, 64.6912,154.63}**.
- Enter ***xsqu*:=*x*²**
- Enter $\boldsymbol{k := \dfrac{y}{xsqu}}$



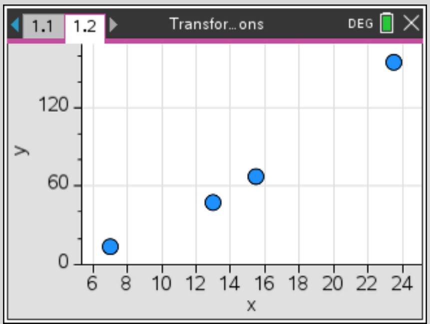
Answer: x^2 values are 49, 169, 240.25, 552.25; $k = 0.28$;
Equation that links y and x is $y = kx^2 = 0.28x^2$.

... continued

Linearising scatter plots (continued)

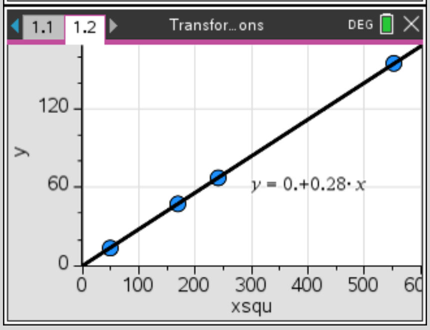
To display a scatter plot, add a **Data & Statistics** page, then:

- Press tab to activate **Click to add variable** underneath the horizontal axis and select the variable ***x***.
- Press tab to activate **Click to add variable** to the left of the vertical axis and select the variable ***y***.



To display a linearised scatter plot and add a least squares regression line:

- Click on ***x*** variable and select the variable ***xsqu***.
- To show the least squares regression line, press menu **> Analyse > Regression > Show Linear (a + bx)**

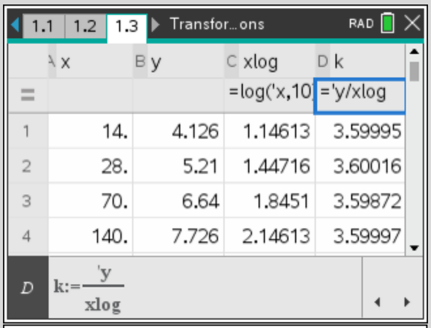


This will display a 'linearised' scatter plot of y against x^2, with the least squares line drawn over the plot. Click on the line to display the equation of the least squares regression line.

***Note**: Remember the calculator is showing a linear relationship between y and the square of x, so the correct equation is* $y = 0.28x^2$.

(b) On a **Lists & Spreadsheet** page, assign x and y as follows:

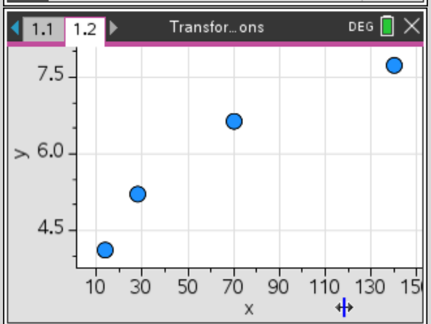
- In the column A heading cell, enter the name ***x***.
- In the column B heading cell, enter the name ***y***.
- Enter the data for ***x*** and ***y*** in columns A and B respectively.
- In the column C heading cell, enter the name ***xlog***.
- In the column D heading cell, enter the name ***k***.
- In the column C formula cell, enter **=log('*x*,10)**.
- In the column D formula cell, enter **='*y*/*xlog***.



Answer: The $\log_{10}(x)$ values are approximately 1.146, 1.447, 1.845, 2.146; k is approximately 3.6; An equation that links y and x is $y = k\log_{10}(x) = 3.6\log_{10}(x)$.

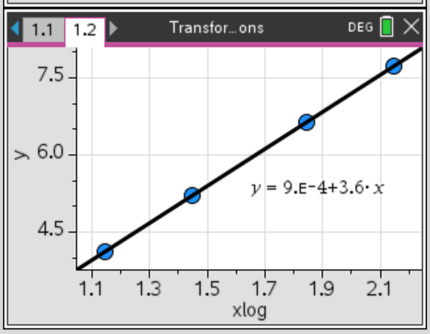
To display a scatter plot, on the **Data & Statistics** page:

- Click underneath horizontal axis and select ***x***.
- Click to the left of vertical axis and select ***y***.



To display a linearised scatter plot and add a least squares regression line, add a **Data & Statistics** page, and then:

- Click underneath horizontal axis and select ***xlog***.
- To show the least squares regression line, press menu **> Analyse > Regression > Show Linear (a + bx)**.



This will display a 'linearised' scatter plot of y against $\log_{10}(x)$, with the least squares line drawn over the plot. Click on the line to display the equation.

***Note**: The calculator is showing a linear relationship between y and the logarithm of x, so the correct equation is* $y = 3.6\log_{10}(x)$. *The y-intercept displayed is 9E–4* $= 9 \times 10^{-4}$ (≈ 0).
The display precision can be modified by pressing menu ***> Settings > Display Digits***.

Solving to find a non-linear model

There are three further models we can use to analyse non-linear relationships:

$$y = kx^2 + c \qquad y = \frac{k}{x} + c \qquad y = k\log_{10}(x) + c$$

The k value will represent the gradient of the linearised data values, and the c value represents a vertical translation which we see as the y-intercept (i.e. the point $(0,\ c)$ on the linearised graph).

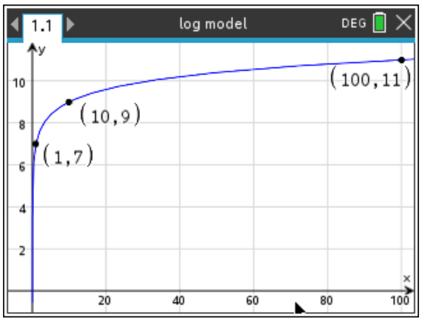
(a) A non-linear model of the form $y = \frac{k}{x} + c$ is known to perfectly fit the data set shown.

Find the values of k and c, and plot the graph of the equation showing the points in the table.

x	1	2	4	10
y	9	7	6	5.4

(b) The graph shown is of the form $y = k\log_{10}(x) + c$.

Find the values of k and c.



(a) To find the values of k and c, select any two of the coordinate pairs (x, y) and substitute into $y = \frac{k}{x} + c$.

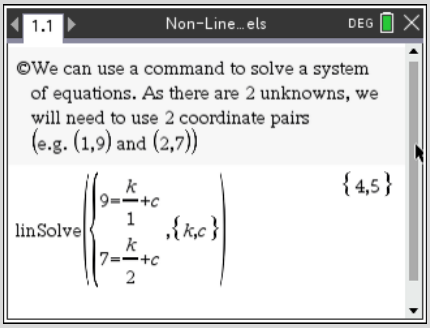
For this example, $(1, 9)$ and $(2, 7)$ will be used.

At $(1, 9)$: $9 = \frac{k}{1} + c$ … [1] and

At $(2, 7)$: $7 = \frac{k}{2} + c$ … [2]

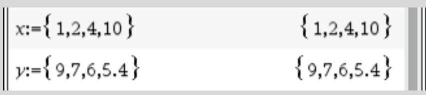
To solve these equations for k and c, on a **Calculator** page:

- Press menu > **Algebra > Solve System of Equations > Solve System of Linear Equations.**
- In the dialog box that follows:
  - For **Number of equations**, select **2**
  - For **Variables**, enter $\boldsymbol{k, c}$, then click **OK**.
- Enter equations [1] and [2] as shown.



Answer: $c = 5$, and $k = 4 \ \Rightarrow y = \frac{4}{x} + 5$.

To plot the four points, first enter them as a set of x and y values:



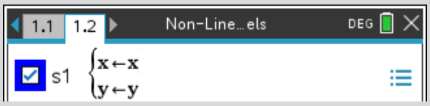
- Press ctrl [:=] to access the **Assign** [:=] command.
- For x, enter $\boldsymbol{x}$**:={1,2,4,10}**.
- For y, enter $\boldsymbol{y}$**:={9,7,6,5.4}**.

… continued

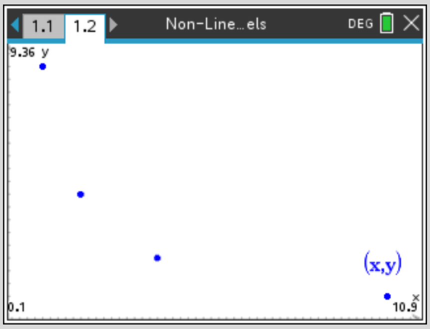
Solving to find a non-linear model (continued)

To plot these values, add a **Graphs** page and then:

- Press [menu] > **Graph Entry/Edit** > **Scatter Plot**
- To define the scatter plot s1
  - For ***x***, enter ***x***.
  - For ***y***, enter ***y***.
- Press [menu] > **Window/Zoom** > **Zoom - Data**



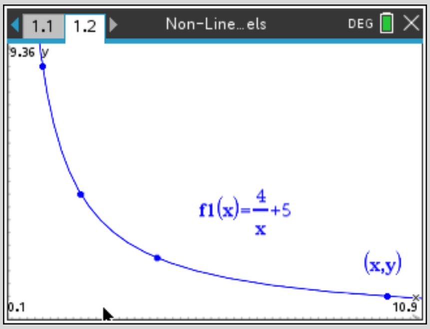
This displays the data points in a suitable window.



To overlay the graph of the equation $y=\frac{4}{x}+5$:

- Press [menu] > **Graph Entry/Edit** > **Function**
- Enter $\boldsymbol{f1(x)=\frac{4}{x}+5}$

As shown, this overlays the function with the data points.



(b) To find the values of k and c, select any two of the coordinate pairs (x, y) and substitute into $y=k\log_{10}(x)+c$. For this example, $(10, 9)$ and $(100, 11)$ will be used.

$$\text{At } (10, 9):\quad 9=k\log_{10}(10)+c \;\ldots\; [1]$$

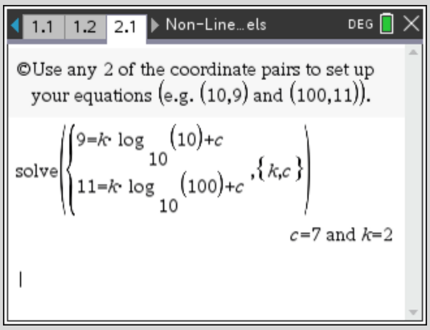
$$\text{At } (100, 11):\quad 11=k\log_{10}(100)+c \ldots\; [2]$$

To solve these equations for k and c, on a **Calculator** page:

- Press [doc▾] > **Insert** > **Problem** and add a **Calculator** page

Note: *Using* [doc▾] ***> Insert > Problem*** *creates a fresh problem space in the current document where any previously defined variables and functions do not apply. Alternatively, creating a new document will achieve a similar result.*

- Press [menu] > **Algebra** > **Solve System of Equations** > **Solve System of Equations.**
- In the dialog box that follows:
  - For **Number of equations**, select **2**
  - For **Variables**, enter $\boldsymbol{k,c}$, then click **OK**.
- Enter equations [1] and [2] as shown.



Answer: $c=7$, and $k=2 \;\Rightarrow y=2\log_{10}(x)+7$.

2.3. Space, measurement and applications of trigonometry

2.3.1. Unit conversions

Using the Conversion Assistant for unit conversions

Convert the following measurements into the units given in brackets.

(a) 2 m^2 (cm^2)
(b) 300 ml (cm^3)

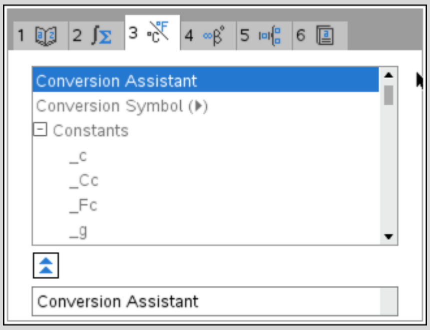
(a) To convert 2 m^2 to cm^2, on a **Calculator** page:

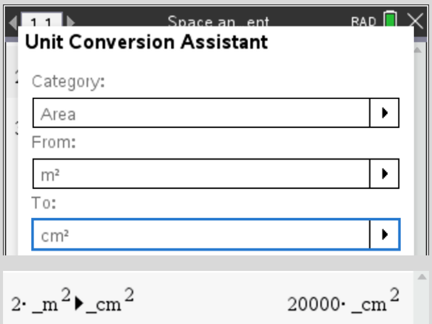
- Type **2**
- Press [📖] [3] > **Conversion Assistant**

In the dropdown dialog box that follows, select the following:

- Category: **Area**
- From: $\mathbf{m^2}$
- To: $\mathbf{cm^2}$

- Click **OK** to close the dialog box, then press [enter].





Answer: 20000 cm^2

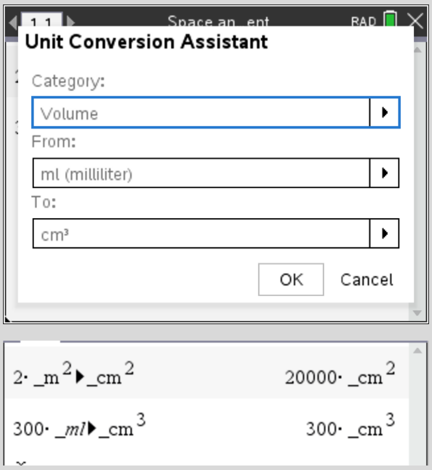
(b) To convert 300 ml to cm^3, on a **Calculator** page:

- Type **300**
- Press [📖] [3] > **Conversion Assistant**

In the dropdown dialog box that follows, select the following:

- Category: **Volume**
- From: **ml**
- To: $\mathbf{cm^3}$

- Click **OK** to close the dialog box, then press [enter].



Answer: 300 cm^3

2.3.2. Scientific notation, rounding and significant figures

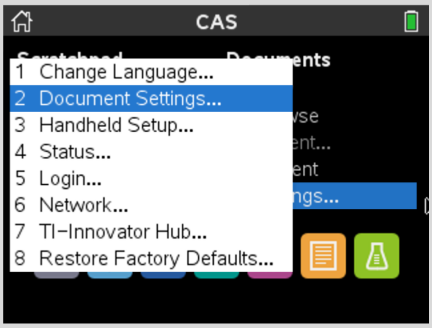
Displaying a particular number of digits and rounding to a certain number of decimal places

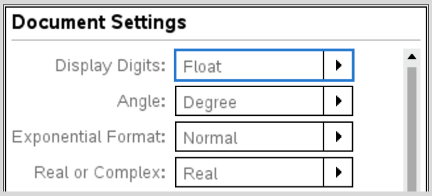
The calculator can be set to display a particular number of digits in the answers.

Calculate $\frac{28.1\times 3.67}{5.2}$, correct to 2 decimal places.

To set the number of display digits:

- From any page, press [on] > **Settings** > **Document Settings** (or alternatively, click the battery icon).
- In the dropdown dialog box, select:
  - Display Digits: **Float**



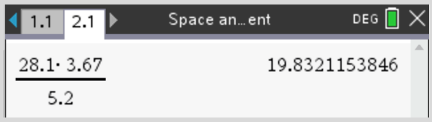


***Note:** From the dropdown dialog box, you can select a **Float** setting through to **Float 12**. This gives the number of digits to be displayed. **Float** displays up to 12 digits but adjusts to the calculation. The factory default setting is **Float 6**.*

To calculate $\frac{2.81\times 3.67}{5.2}$, on a **Calculator** page:

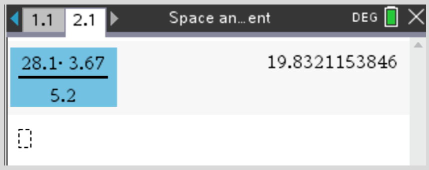
- Enter **2.81** [×] **3.67** [÷] **5.2** into the numerator.

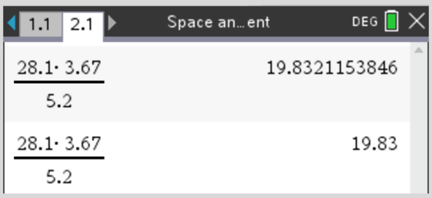
This **Float** setting returns a 12-digits output: 19.8321153846. To round this result to 2 decimal places, 4 digits need to be displayed.



Approach 1. Adjust the **Float** value.

- Press [on] > **Settings** > **Document Settings**, then select:
  - Display Digits: **Float 4**
- Click **OK** to close the dialog box.
- Press the up-arrow key ▲ twice then press [enter]. This pastes the calculation into the entry line.
- Press [enter] to complete the calculation in **Float 4** setting.



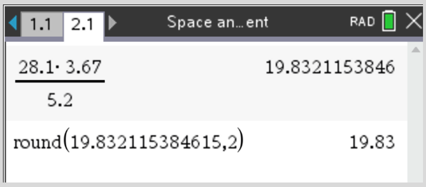


Answer: 19.83, correct to 2 decimal places.

Approach 2. Use the **round** command.

With the Display Digits settings still in **Float**:

- Press [book] [1] [S], navigate to **round** and press [enter].
- Enter **round(Ans,2)** by pressing [ctrl] [(−)] for **Ans**. (Alternatively, press ▲[enter] to paste the answer.) The syntax is **round(expression,decimal places)**



Answer: 19.83, correct to 2 decimal places.

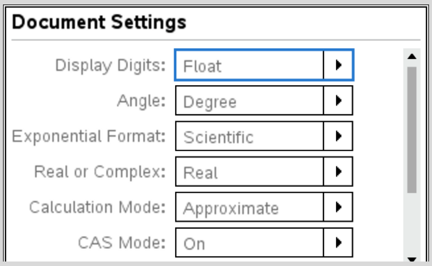
Using scientific notation

Calculations can also be carried out in scientific notation format.

(a) Evaluate 28319×45610. Give the answer in scientific notation.
(b) Calculate $7.33\times10^{-5}\div8.6\times10^{7}$.

To set **Scientific** notation format, from any page:

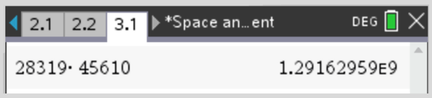
- Press [⌂ on] > **Settings** > **Document Settings**, then select:
  - **Exponential Format**: Scientific
  - **Calculation Mode**: Approximate
- Click **OK** to close the dialog box.



(a) To evaluate 28319×45610, on a **Calculator** page:

- Enter **28319 × 45610**

Answer: $1.2916...\times10^{9}$, displayed as 1.29162959E9.

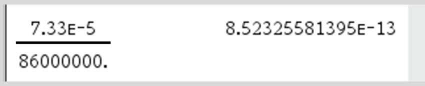


Note. *The "E9" represents the exponent of base 10.*

(b) To calculate $7.33\times10^{-5}\div8.6\times10^{7}$:

- Enter **7.33E–5 ÷ 8.6E7** by pressing [EE] for **E–5** and **E7**.

Answer: $8.523...\times10^{-13}$, displayed as 8.5232558139E−13.



Note: *The [10ˣ] key can be used in place of the [EE]. However, brackets or the fraction template are required for the divisor.*

Exploring significant figures

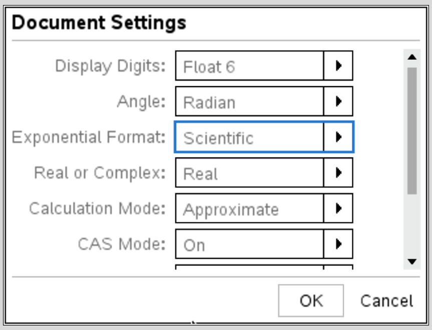
Note: *Significant figures are used in scientific and other measurements to indicate the accuracy of the measurement, which will depend on the precision of the device used to make the measurement. A significant figure is a digit in a number that contributes to indicating its accuracy.*

(a) A curtain fabric is advertised as being 1.8 m wide. When measured to the nearest mm, it is found to be 1826 mm wide.
  (i) State this measurement in mm, correct to 4, 3, 2 and 1 significant figures.
  (ii) Express 1826 in scientific notation of the form $m\times10^{n}, 1\le m<10$. Hence confirm the answer to part (i) above by rounding m to display 4, 3, 2 and 1 significant digits.

(b) Rolls of the curtain fabric are 9.7 m long, measured to the nearest tenth of a metre, and 1.826 m wide, measured to the nearest mm. Determine the total area of 8 rolls of the fabric, in m², correct to the maximum number of significant figures based on these measurements.

(a) (i) Answer.

Sig Fig	**Width** (mm)	**Nearest**	**Precision** (mm)
4	$1826 = 1.826\times10^{3}$	mm	1825 < w < 1827
3	$1830 = 1.83\times10^{3}$	cm	1820 < w < 1840
2	$1800 = 1.8\times10^{3}$	dm	1700 < w < 1900
1	$2000 = 2\times10^{3}$	m	1000 < w < 3000

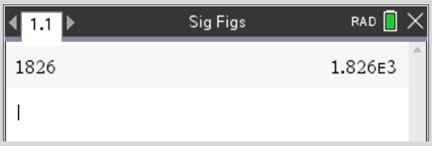


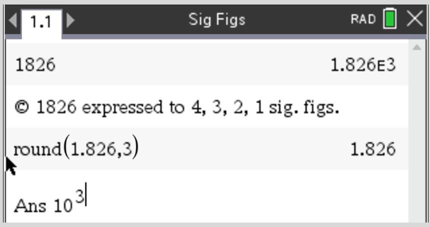
... continued

Exploring significant figures (continued)

(ii) To express 1826 in **Scientific** notation:

- Click the **battery icon** (top right corner) and select **Document Settings**. In the dropdown dialog box select: Display Digits: **Float 6**, Exponential Format: **Scientific**.
- Enter **1826**.
- Restore **Document Settings** to Exponential Format: **Normal**.

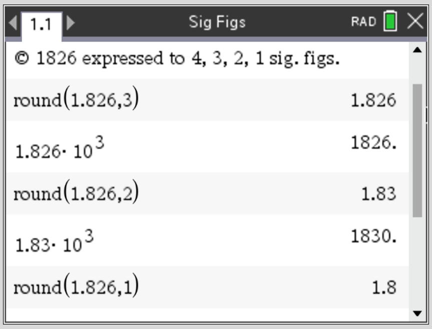


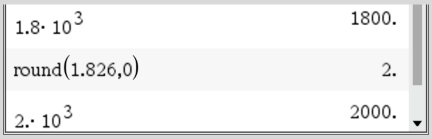


Answer: 1.826×10^3, displayed as $1.826\text{E}3$.

To round 1.826 to display 4, 3, 2 or 1 significant figures:

- Press [catalog] [1] [S], navigate to **round** and enter **round(1826,3)**. The syntax is **round(**expr,decimal places**)**
- Press [ctrl] [(−)] ([ans]), then [10ˣ] to enter **Ans 10³**.
- Similarly, enter **round(1826,2)** followed by **Ans×10³** , enter **round(1826,1)** followed by **Ans×10³**, and enter **round(1826,0)** followed by **Ans×10³**, as shown.





Answer: 4 sig. figs.: $1.826\times10^3 = 1826$,

3 sig. figs.: $1.83\times10^3 = 1830$, 2 sig. figs.: $1.8\times10^3 = 1800$

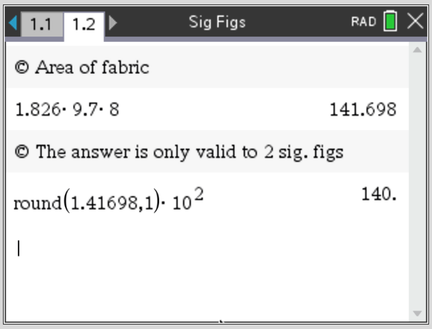
1 sig. fig.: $2\times10^3 = 2000$

Notes:
1. *The trailing zeros are not significant figures.*

2. *Writing a measurement in scientific notation clarifies significant figures by eliminating trailing and leading zeros.*

(b) To determine the total area of the 8 rolls of fabric:

- Enter **1.826×9.7×8**



Notes:
1. *The **Float 6** setting displays the result as **141.698**. However, when multiplying or dividing, the rule is to limit significant figures in the final answer to the smallest number of significant figures in the original numbers, i.e. 2.*

2. *The number **8** in the calculation is an **exact** count and should be regarded as having infinite accuracy.*

3. *Although the calculation returns a 6-digit result in **Float 6** setting, it should not be assumed that the final answer is accurate to 6 significant figures.*

To determine the area, correct to 2 significant figures (i.e. 1 decimal place in scientific notation):

- Enter **round(1.41698,1)×10²**, using the [10ˣ] key.

Answer: $140\text{ m}^2 = 1.4\times10^2\text{ m}^2$.
The value of the area is only valid to 2 significant figures due to the accuracy of the length (9.7 m, 2 significant figures).

2.3.3. Measurement Formulas

Defining a formula

(a) Calculate the volume of cylinders with the following dimensions, to 2 decimal places:
- **(i)** radius is 5.2 m and the height is 12.3 m.
- **(ii)** radius is fixed at 15 m and the height is 2.25 m, 4.5 m or 6.75 m.

(b) Calculate the radius of a sphere given the following volumes, to 2 decimal places.

(i) $140\,\text{cm}^3$ **(ii)** $250\ \text{cm}^3$ **(iii)** $380\ \text{cm}^3$

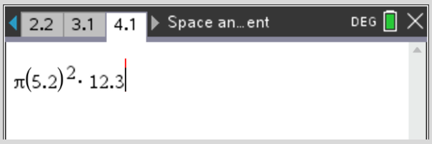
(c) Determine the volume of a right circular cone with the following dimensions.
- **(i)** Radius is 7.2 cm and the height is 14 cm. Give the answer to one decimal place.
- **(ii)** Radius is 14 cm and the height is 7.2 cm. Give the answer to one decimal place.

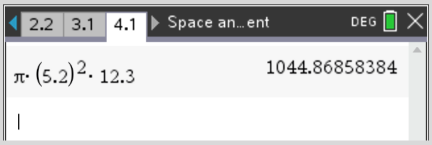
(a)(i) To calculate the volume of a cylinder with $r = 5.2\,\text{m}, h = 12.3\,\text{m}$, on a **Calculator** page:

- Press [π▸] and select π .
- Enter $\boldsymbol{\pi(5.2)^2 \times 12.3}$, by using the [x²] key.

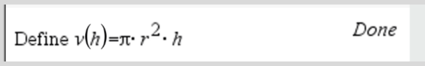
Note: Alternatively, enter π by pressing the letters [P][I].

Answer: $1044.87\,\text{m}^3$(2 decimal places).



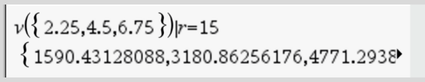


(a)(ii) To define a function for the volume of a cylinder formula, on a **Calculator** page:
- Press [menu] > **Actions** > **Define**.
- Enter **Define** $\boldsymbol{v(h) = \pi \times r^2 \times h}$, as shown.



To calculate multiple values of the volume, $\boldsymbol{v}$, using the set brackets (braces) to create a list of values of $\boldsymbol{h}$:

- Type the expression, ***v*****({2.25,4.5,6.75})** by pressing [ctrl][)] to insert the pair of braces.
- Move the cursor to the right of the last bracket.
- Press [ctrl][=], select '|', and enter ***v*****({2.25,4.5,6.75})|*****r*****=15**

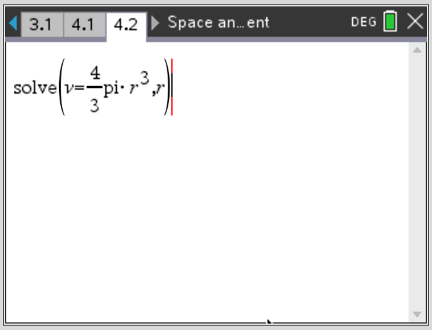


Answer: $1590.43\,\text{m}^3$, $3180.86\,\text{m}^3$, $4771.29\,\text{m}^3$

(b) A measurement formula can be stored in a variety of ways when multiple calculations are made with the same formula.

To rearrange the volume of a sphere formula and make the radius the subject, with **Document Settings** set to Calculation Mode: **Auto**, on a **Calculator** page:

- Press [menu] > **Algebra** > **Solve**
- Enter $\textbf{solve}\left(\boldsymbol{v = 4/3 \times \pi \times r^3, r}\right)$.



***Note:** If the **Document Settings** are in **Approximate** Mode, then the result is displayed with decimal approximations.*

... continued

Defining a formula (continued)

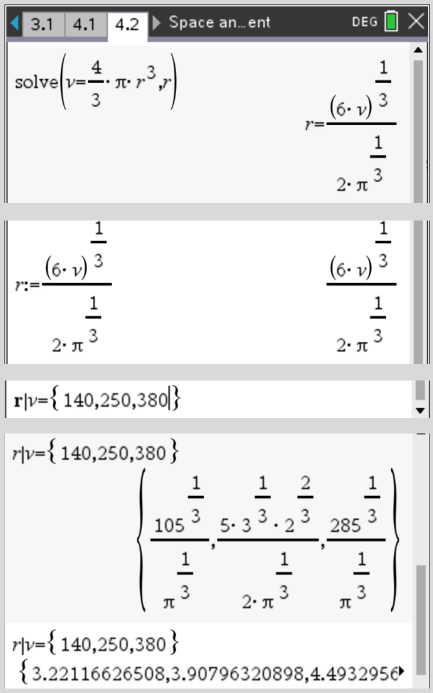
- Enter ***r*:=** to define a function for the radius, pressing ctrl ⊞ for the **assign** symbol, :=.
- Press the up-arrow key, ▲ to select the answer line above and press enter.
- Edit the line to remove "***r*** =", as shown.
- Press enter to store ***r*** as a function of ***v***.

To calculate multiple values of the radius, ***r***, for spheres with volumes, 140 cm^3, 250 cm^3 and 380 cm^3, use the set brackets (braces) to create a list of values of ***v***, as follows:

- Enter ***r*|*v* ={140,250,380}**.

*Note: Depending on the **Document Settings**, the calculation may return exact answers. Pressing* ctrl enter *will convert exact answers to decimal approximations.*

Answer: (i) 3.22 m (ii) 3.91 m (iii) 4.49 m (2 decimal places).

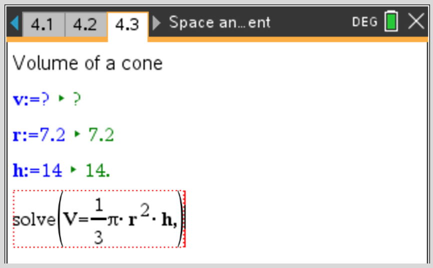


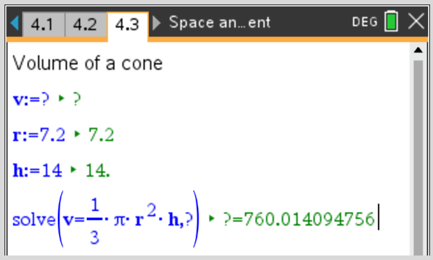
(c) (i) To set up an editable template for the volume of a cone with $r = 7.2$ cm and $h = 14$ cm, on a **Notes** page:

- Enter the page heading, **Volume of a cone**.
- Press ctrl M to insert a **Maths Box** and then
  - Enter $V := ?$
  - Enter $r := 7.2$
  - Enter $h := 14$
  - Enter $\textbf{solve}\left(V = \dfrac{1}{3}\pi \times r^2 \times h, ?\right)$.

Note: Press ctrl ⊞ (:=) *for **assign** symbol*. Press ?!▸ to select the **?** *symbol. The solve command can be typed or found in* menu ***> Calculations > Algebra > Solve***.

Answer. 760.0 cm^3 (1 decimal place)

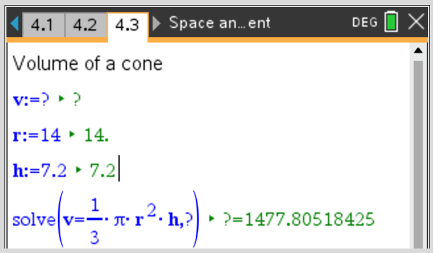




Note: This method can be used for solving most formulas in the General Mathematics course.

(ii) Edit the entries to ***r* :=14** cm and ***h*:=7.2** cm to calculate the new volume. You will notice the volume calculation changes "dynamically" with each change in the inputs.

Answer. 1477.8 cm^3 (1 decimal place)



Creating a user defined function to calculate the volume and surface area of a cylinder

A large cylindrical milk vat has a diameter of 3.6 m and a height of 3.2 m.

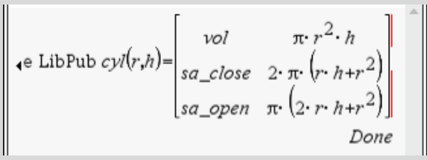
(a) Create a User Defined function to calculate the volume and surface area of a closed and open cylinder, given its radius and height. Save the function so that it is accessible in the Catalogue.

(b) Use the User Defined function from part (a) above to determine the volume and surface area of the vat, with and without a lid, correct to one decimal places.

(c) Given that 1 m^3 is equivalent to 1000 litres, determine how many litres of milk the vat can hold, giving the answer to the nearest hundred.

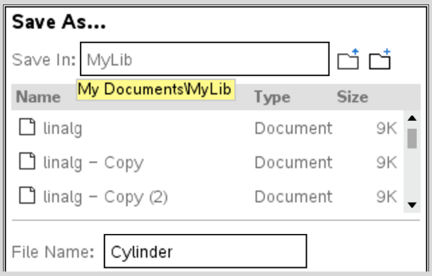
(a) To set up a **User Defined** function, on a **Calculator** page:

- Press [menu] > **Actions** > **Library** > **Define LibPub (…)**.
- Enter the function definition, as shown, pressing [matrix template key] to select a 3 × 2 matrix template.



To save the function so that it is accessible in the **Catalogue**:

- Press [doc▾] > **File** > **Save As …**
- Select the folder **MyLib**, name the file, then click **Save**.

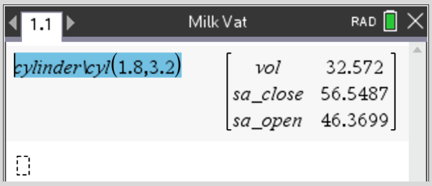


To refresh the **Catalogue** and make the function available:

- Press [menu] > **Actions** > **Library** > **Refresh Libraries**.
- Open a new document to test that the function is available.

(b) To find volume and surface area of vat with $d = 3.6$ m and $h = 3.2$ m, on a **Calculator** page:

- Press [catalogue] [6]. Select the **User Defined** function, **cylinder.**
- Enter **cylinder\cyl(1.8,3.2)**, for $r = 1.8$ m, $h = 3.2$ m.



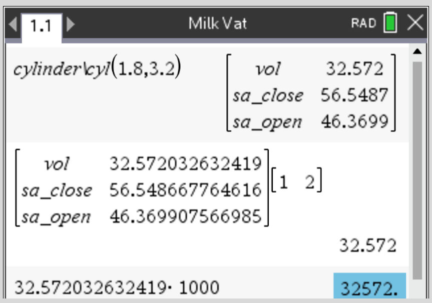
Answer. $V = 32.6$ m^3. Closed vat, $SA = 56.6$ m^2.
Vat with lid removed, $SA = 46.4$ m^2 (1 decimal place).

(c) To extract the volume value from the matrix:

- Press [ctrl] [(–)] (ans) and enter **Ans[1,2]**. The syntax is **Ans[*row number*, *column number*]**.

To calculate the capacity of the vat in litres:

- Enter **Ans×1000**.



Answer. The vat can hold 32,600 litres of milk (to the nearest hundred).

Applying a user defined function to explore the effect of a scale factor on volume and area

The large cylindrical vat in the previous problem has a diameter of 3.6 m and a height of 3.2 m. Consider another vat with the same proportions, but which is a smaller than the large vat by a scale factor of $\frac{1}{2}$. Use the User Defined function from the previous problem to determine the following.

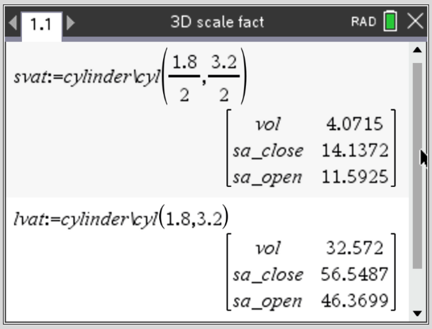
(a) The ratio of the total surface areas of the large vat to that of the smaller vat.

(b) The ratio of the volume of the large vat to that of the smaller vat.

(c) Comment on the reasons for the results in parts (a) and (b) above.

To find the volumes and surface areas, on a **Calculator** page:

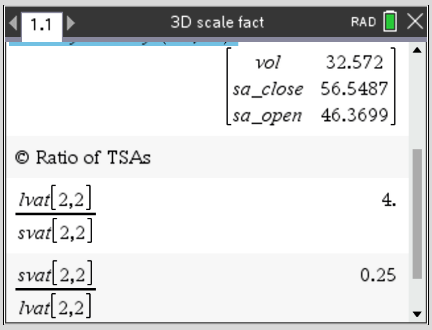
- Type, ***svat*:=**, then press [📖][6] and select the **User Defined** function, **cylinder**. (Press [ctrl][⊞] (:=) for **assign**.)
- Enter ***svat*:=cylinder\cyl(1.8/2,3.2/2)** for the smaller vat.
- Key in, ***lvat*:=**, then press [📖][6] and select the **User Defined** function, **cylinder**.
- Enter ***lvat*:=cylinder\cyl(1.8,3.2)** for the large vat.



(a) To determine the ratio of the total surface areas:

- Enter ***lvat*[2,2]/ *svat*[2,2]** by pressing [var] to select variables ***lvat*** and ***svat***. The syntax for selecting an element from the matrix is **[*row number*,*column number*]**.
- Similarly, enter ***svat*[2,2]/ *lvat*[2,2]**.

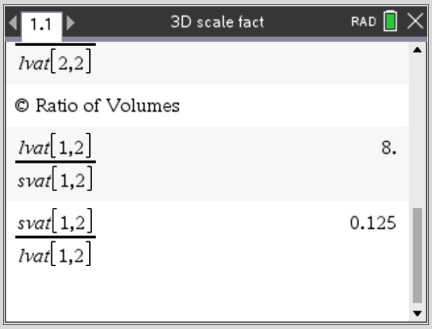
Answer. If the scale factor is 2, then the surface area is 4 times as large. Conversely, if the scale factor is $\frac{1}{2}$, then the ratio of the surface areas is $0.25 = \frac{1}{4}$.



(b) To determine the ratio of the volumes:

- Enter ***lvat*[1,2]/ *svat*[1,2]** by pressing [var] to select variables ***lvat*** and ***svat***.
- Similarly, enter ***svat*[1,2]/ *lvat*[1,2]**.

Answer. If the scale factor is 2, then the volume is 8 times as large. Conversely, if the scale factor is $\frac{1}{2}$, then the ratio of the volumes is $0.125 = \frac{1}{8}$.



(c) Answer. Surface area requires the scale factor be applied in 2 dimensions. If $k = 2$, then TSA increases by $2 \times 2 = 4$. Volume requires the scale factor be applied in 3 dimensions. If $k = 2$, then volume increases by $2 \times 2 \times 2 = 8$.

2.3.4. Trigonometry

Evaluating the trigonometric ratios

(a) Calculate the following. Give the answers to parts (i) and (ii) to four decimal places and the answer to part (iii) to two decimal places.

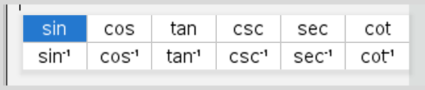
(i) $\sin(15.28^\circ)$ **(ii)** $\cos(78^\circ 14')$ **(iii)** $\dfrac{15\sin(37^\circ)}{\sin(46^\circ)}$

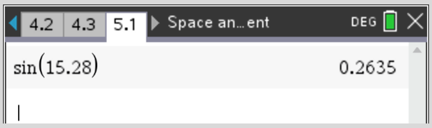
(b) **(i)** Convert $28^\circ 29' 48''$ to decimal degrees. Give the answer to four decimal places.

(ii) Convert 132.16° to degrees, minutes and seconds.

(c) Given $36.5\tan(\theta) = 82.3$, where θ is an internal angle of a right-angled triangle, find the value of θ. Give your answer to the nearest tenth of a degree.

(a) (i) To evaluate $\sin(15.28^\circ)$, on a **Calculator** page:

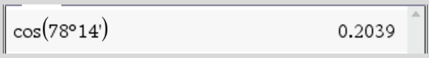




- Press [⌂ on] > **Document Settings** (alternatively, click the battery icon at the top right corner of the screen).
- In the dropdown dialog box, select as follows. Display digits: **Float 4**, Angle: **Degree**, Calculation Mode: **Approximate**.
- Press [trig] and select **sin**.
- Enter **sin(15.28)**.

Answer: 0.2635.

(ii) To evaluate $\cos(78^\circ 14')$:

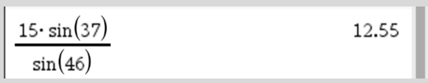


- Press [trig] and select **cos**.
- Press [template] then select the **dd°mm′ss.ss″** template.
- Enter **cos(78°14′)**, as shown.

Answer. 0.2039

Notes:
1. *Degrees and Minutes symbols can also be selected by pressing [π▸] or [?!▸] or [ctrl][📖] or [📖][1] (symbols at the top of the* ***Catalogue****).*
2. *If the* ***Document Settings*** *are set to* **Calculation Mode: Auto**, *you may get an answer in terms of another trigonometric function. Press [ctrl][enter] to convert these answers to decimal approximations.*

(iii) To evaluate $\dfrac{15\sin(37^\circ)}{\sin(46^\circ)}$, on a **Calculator** page:



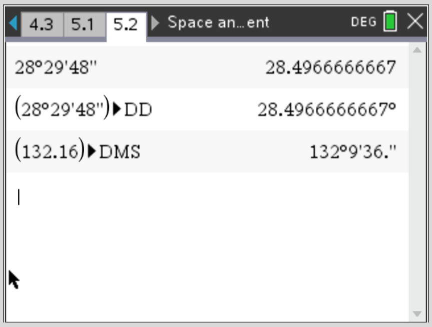
- Enter **15sin(37)/sin(46)** (pressing [trig] to select **sin**)

Answer: 12.55

... continued

Evaluating the trigonometric ratios (continued)

(b) (i) To convert $28°29'48''$ to decimal degrees (with **Document Settings** set to Display digits: **Float)**, on a **Calculator** page:



- Press [template] and select the **dd°mm′ss.ss″** template.
- Enter **28°29′48″**, pressing [tab] to move between fields.

Answer: $28.4967°$ (four decimal places).

__Note:__ The command **▸DD** *can be entered manually by pressing* [ctrl] [catalog] *to select the conversion symbol,* ▸*, then typing* **DD**. *Alternatively, use the* ***Decimal Degrees*** *conversion by pressing* [catalog] [D] *and selecting* **▸DD**.

(ii) To convert $132.16°$ to degrees, minutes and seconds:

- Type **132.16**, press [catalog] [1] [D] and select **▸DMS**. Press [enter].

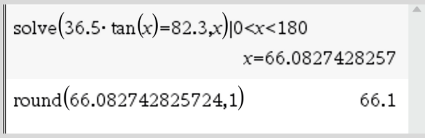
Answer: $132°9'36''$.

__Note__: Alternative approach to enter **▸DMS**: *press* [ctrl] [catalog] *and select the conversion symbol,* ▸*, then type* **DMS**.

(c) To solve $36.5\tan(\theta) = 82.3$ for $0° < \theta < 90°$, on a **Calculator** page:

- Press [menu] > **Algebra** > **Solve**.
- Type $\textbf{solve}(\textbf{36.5tan}(x) = \textbf{82.3}, x)$, pressing [trig] to select **tan**. (Do **not** press [enter] yet).

To set a domain restriction, before pressing the final [enter]:



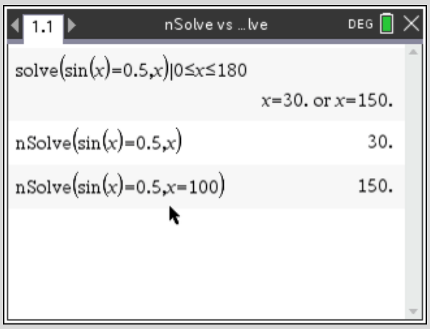
- Press [ctrl] [=] to select the **given** symbol, |.
- Enter domain, $\mathbf{0 < x < 180}$, pressing [ctrl] [=] to select <.

To round to the nearest tenth of a degree (1 decimal place):

- Press [catalog] [S] and select **round**.
- Press the up-arrow key, ▲, then [enter] to paste the answer.
- Delete the ***x*** =.
- Enter **round(66.082742825724,1)**

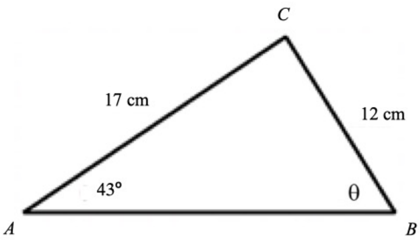
Answer: $66.1°$ to the nearest tenth of a degree.

Notes:
(1) *Using* ***x*** *instead of* $\boldsymbol{\theta}$ *is suggested. Press* [π▸] *or* [ctrl] [catalog] *for* $\boldsymbol{\theta}$.
(2) *To solve for a particular angle, it is necessary to restrict the domain and avoid a general solution.*
(3) *An alternative method is to use the* ***nsolve*** *command, which attempts to find the nearest solution to x = 0. It only returns a single solution, and its command syntax is* ***nSolve(Equation,variable)***. *It is also possible to find a solution by giving a 'guess' at the solution. For example,* ***nsolve(sin(x)=0.5,x=100)*** *will find the solution x = 150°).*



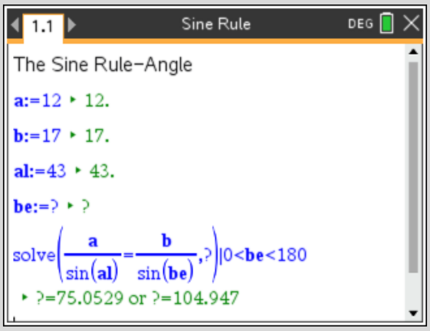
Solving trigonometric application questions

(a) Find the angle, θ, in the triangle shown, to two decimal places.



(b) The triangle above is one of two possible triangles with measurements, $A = 43°$, $a = 12$, $b = 17$.
Use the sine rule to determine the number of triangles that can be represented given:
(i) $A = 120°$, $a = 250$, $b = 195, B = ?$ **(ii)** $A = 42°$, $a = 3$, $b = 8, B = ?$

(a) We can approach this Sine Rule question by setting up a **Notes** page that can be reused for all Sine Rule angle calculations.

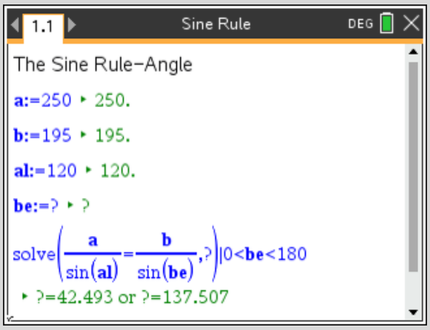


To set up an interactive Sine Rule template, on a **Notes** page:

- Enter the heading, **The Sine Rule – Angle**.
- Press ctrl M to insert a **Maths Box** and then:
  - Enter **a := 12**
  - Enter ***b*** **:= 17**
  - Enter ***al*** **:= 43**
  - Enter ***be*** **:= ?**
  - Enter $\textbf{solve}\left(\dfrac{a}{\sin(al)} = \dfrac{b}{\sin(be)}, ?\right) \mid 0 < be < 180$

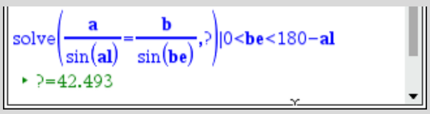
Answer: $75.0529..., 104.947....$ $(\theta = 75.05° \text{ or } 104.95°)$.

Notes:
1*. This is an example of the Ambiguous case for a triangle, hence there are two possible solutions for the angle.*
2*. When finding the angle, you need to restrict the domain to avoid general solutions. Alternatively, use* ***nSolve****.*

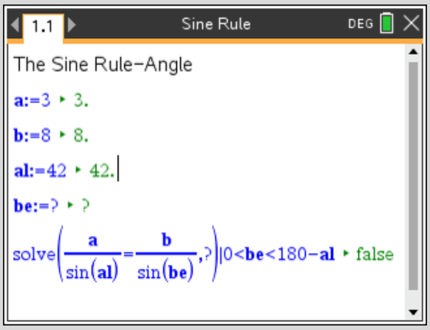


(b)(i) To solve the triangle $A = 120°$, $a = 250$, $b = 195, B = ?$:

- Edit the Sine Rule template from part (a), as shown.



Answer. Only **one** triangle is possible, with $B \approx 42.49°$. $B \neq 137.5...°$ because $120° + B < 180°$. This can be seen if the domain is restricted to $\mathbf{0 < be < (180 - al)}$.



(b)(ii) To solve the triangle $A = 42°$, $a = 3$, $b = 8, B = ?$:

- Edit the Sine Rule template from part (a), as shown.

Answer. **No** triangle is possible, hence the **false** result.

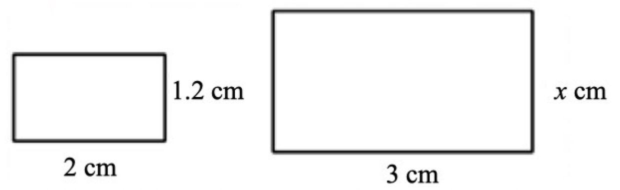
Note: *This approach can be used to also solve Sine Rule for side lengths, Cosine Rule for angles or side lengths and Area of triangle calculations. These Notes pages can be saved into* ***My Documents*** *and retrieved for future calculations.*

2.3.5. Similarity and scale factor

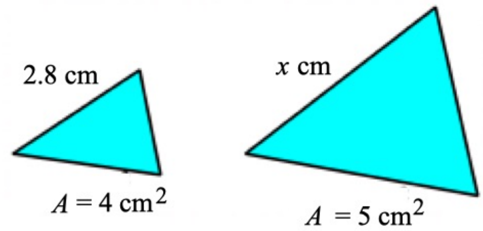
Finding and using the scale factor

For each of the following pairs of similar figures, determine the value of x. Give your answers to parts (b) and (c) to two decimal places.

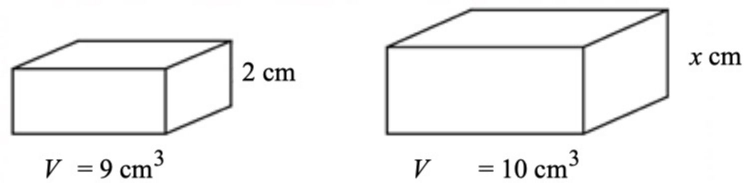
(a)



(b)

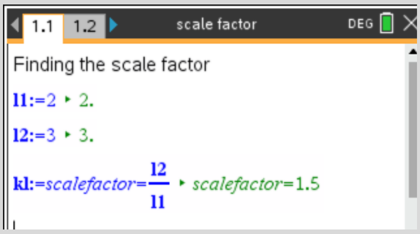


(c)



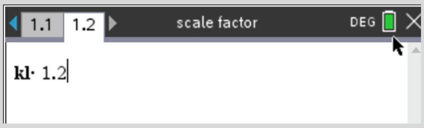
We can approach this question and many other similar questions by setting up a **Notes** page. This can be reused for all similarity calculations requiring the linear scale factor.

(a) We know that the linear scale factor is given by $k = \dfrac{\textit{length of image}}{\textit{length of object}}$



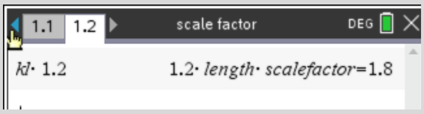
To set up a Scale Factor template that can be edited, on a **Notes** page:

- Enter the heading, **Finding the scale factor**.
- Press ctrl M to insert a **Maths Box** and then
  - Enter $\boldsymbol{l1 := 2}$
  - Enter $\boldsymbol{l2 := 3}$
  - Enter $\boldsymbol{kl := scalefactor = \dfrac{l2}{l1}}$.



To find $\boldsymbol{x}$ using the scale factor $\boldsymbol{kl}$, on a **Calculator** page:

- Press var and select $\boldsymbol{kl}$ (or type $\boldsymbol{kl}$).
- Enter $\boldsymbol{kl \times 1.2}$.

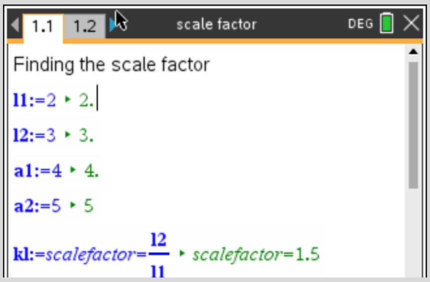


Answer: 1.8 cm

(b) If the linear scale factor for two similar shapes is k, then the area scale factor is given by k^2.

To add a scale factor based on areas, on your **Notes** page:

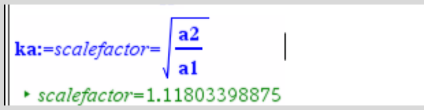
- Press ctrl M to add more **Maths Boxes**, then
  - Enter $\boldsymbol{a1 := 4}$
  - Enter $\boldsymbol{a2 := 5}$



... continued

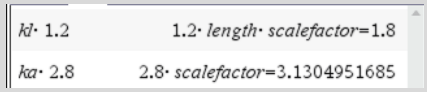
Finding and using the scale factor (continued)

- Enter $ka := scalefactor = \sqrt{\dfrac{a2}{a1}}$



To find x using the scale factor ka, on a **Calculator** page:

- Press [var] and select ka.
- Enter $ka \times 2.8$.

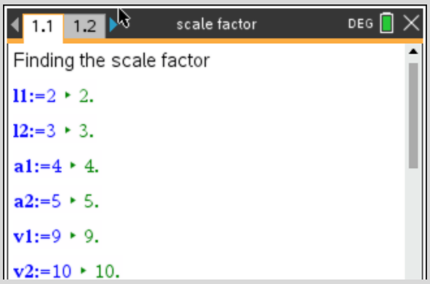


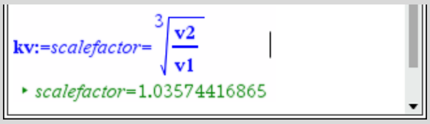
Answer: 3.13 cm (2 decimal places).

(c) If the linear scale factor for two similar objects is k, then the volume scale factor is given by k^3.

To add a scale factor based on volumes, on your **Notes** page:

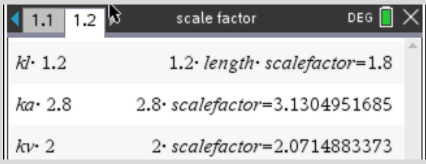
- Press [ctrl] [M] to add more **Maths Boxes**, then
  - Enter $v1 := 9$
  - Enter $v2 := 10$
  - Enter $kv := scalefactor = \sqrt[3]{\dfrac{v2}{v1}}$, pressing [ctrl] [^] (template) for the cubed root template.





To find x using the scale factor kv, on a **Calculator** page:

- Press [var] and select kv.
- Enter $kv \times 2$.



Answer: 2.07 cm (2 decimal places).

Notes:
1*. Ensure that the* ***Document Settings*** *Calculation Mode is* ***Approximate*** *for decimal answers.*
2. *These Notes pages can be saved into* ***My Documents*** *and retrieved for future calculations.*
3*. These scale factors are now defined and can be used in all future calculations.*

2.3.6. Space and measurement in the Geometry application

Exploring perimeter and area of circles, arcs and sectors in the Geometry application

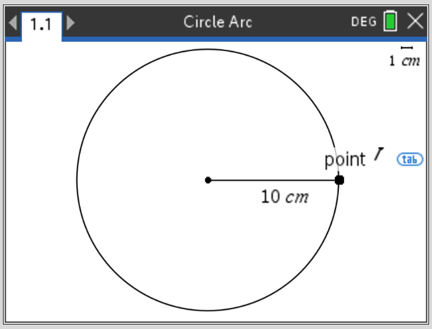
Use Geometry tools to determine the following, correct to one decimal place.

(a) The circumferences and areas of circles with radii of 25, 50 and 75 units.

(b) The arc length and area of the minor sector associated with a circle of radius 50 cm, if the angle subtended at the centre by the arc has magnitude **(i)** 70° **(ii)** 110°.

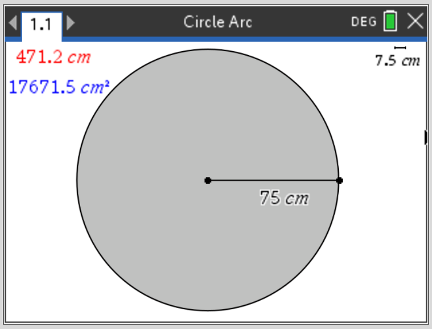
(a) To draw an appropriate circle, on a **Geometry** page:

- Press menu > **Points & Lines** > **Segment**.
- Click two points on the workspace, then press esc to exit.
- Hover over the **segment** and press ctrl menu > **Measurement** > **Length**. Change the length to 10 cm.
- Press ctrl menu > **Pin** to lock-in the **segment** length.
- Press menu > **Shapes** > **Circle**, click each endpoint of the **segment**. Press esc to exit the tool.



To measure the circumference and area:

- Hover over the circle and press ctrl menu > **Measurement** > **Length**. Repeat for **…** > **Area**.
- Press menu > **Settings** > **Display Digits**. Select **Float 6**.
- To make the **radius = 25**, edit the scale (top right-hand corner) to **2.5 cm**. Similarly, edit the scale to **5 cm** for **radius = 50** and to **7.5 cm** for **radius = 75**.
- To round a measurement to one decimal place, hover over the value and press − or + as required.

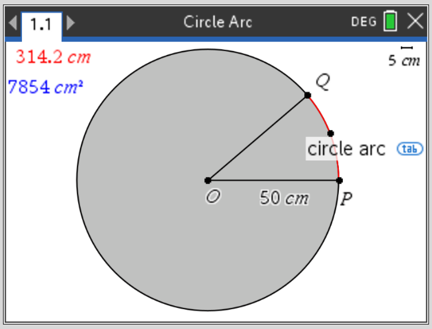


Answer: $r = 25$, $C = 157.1$, $A = 1963.5$

$r = 50$, $C = 314.2$, $A = 7854.0$, $r = 75$, $C = 471.2$, $A = 17671.5$

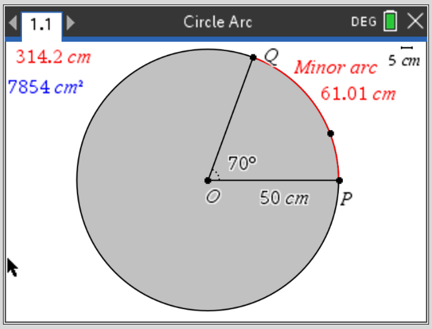
(b) To draw a minor arc that subtends an angle *POQ*:

- Press menu > **Points & Lines** > **Segment**.
- Click the centre of the circle followed by a point on the circumference. Press esc to exit the tool.
- Label the points ***P***, ***O***, ***Q***, as shown, by hovering over a point, pressing ctrl menu > **Label** and entering the label.
- Press menu > **Points & Lines** > **Circle arc**.
- Click the point ***P***, followed by a point on the circle between ***P*** and ***Q***. To complete the arc, click point ***Q***.



To measure angle *POQ* and the associated minor arc:

- Press menu > **Measurement** > **Angle**.
- Click points ***P***, ***O*** and ***Q*** in that order.
- Press menu > **Measurement** > **Length**. Hover over the minor arc and press tab until the label **circle arc** appears.
- Click the arc, then press esc to exit the tool.
- Grab point ***Q*** by hovering over it and pressing ctrl [grab].
- Drag point ***Q*** so that angle ***POQ*** is **(i) 70.0°**, **(ii) 110.0°**.



Answer. **(i)** Angle = 70°, $l = 61.0$ **(ii)** Angle = 110°, $l = 96.0$

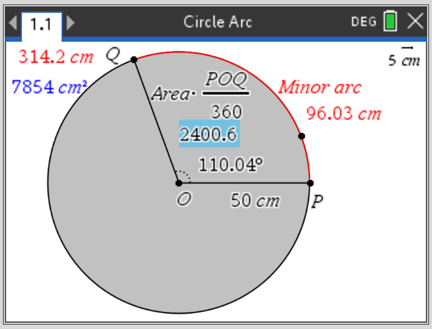
... continued

Exploring perimeter and area of circles, arcs and sectors in the Geometry application (cont.)

To determine the area of the sector using these measurements:

- Press ctrl menu > **Text**.
- In the textbox, enter the expression, ***Area*×*POQ*/360**.
- Press menu > **Actions** > **Calculate**.
- Click the expression, ***Area*×*POQ*/360**. When prompted, click the measurements for **area** and for the angle ***POQ***.
- Drag point ***Q*** to recalculate the area of the minor sector.

Answer. **(i)** 70°, *A* = 1526.4 **(ii)** 110°, *A* = 2400.6



Exploring similarity using Transformation tools in the Geometry application

Consider an arbitrary triangle *ABC*. Let *P* and *Q* be the midpoint of sides *AB* and *BC*, respectively.

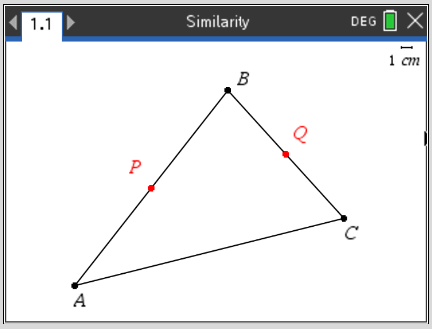
(a) Use the dilation transformation tool to visually verify that triangles *PBQ* and *ABC* are similar, with triangle *ABC* being a dilation of triangle *PBQ* by a scale factor of 2.

(b) Hence use the conditions of similarity to show that the midsegment, *PQ*, is parallel to side *AC* and half its length.

(c) Use Transformation tools to visually verify that the segments joining the midpoints of the triangle *ABC* divide that triangle into four congruent triangles. Hence state the relationship between the scale factor and area of the similar shapes.

(a) To draw the triangle *ABC*, on a **Geometry** page:

- Press menu > **Setting** > **Automatically label points**
- Press menu > **Shapes** > **Triangle**. Click three points on the workspace to form triangle *ABC*.
- Press menu > **Construction** > **Midpoint**. Click on each side *AB* and *BC*, then press esc to exit the tool.
- Label the midpoints as ***P*** and ***Q*** by hovering over the point, pressing ctrl menu > **Label**, and entering label ***P*** then ***Q***.

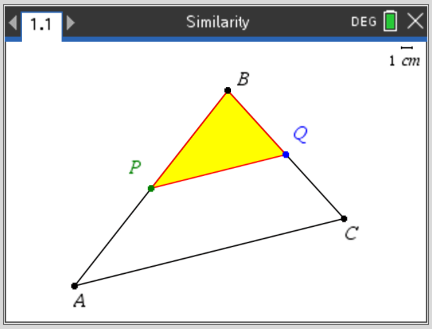


To draw triangle *PBQ*:

- Press menu > **Shapes** > **Triangle**. Click on points ***P***, ***B*** and ***Q***, then press esc to exit the tool.
- Hover over side ***PQ*** and press ctrl menu > **Colour**. Select the desired line and fill colours for triangle ***PBQ***.

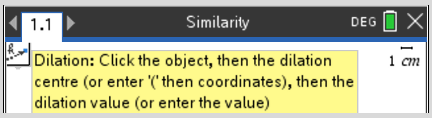
To change the colour of points and labels ***P*** and ***Q***:

- Hover over the point ***P*** and press ctrl menu > **Colour**. Select the desired colour. Repeat for point ***Q***.



To visually verify that triangle ***ABC*** is a dilation of triangle ***PBQ*** using a dilation factor of 2:

- Press menu > **Transformation** > **Dilation**.
- Click triangle ***PBQ***, then press 2 enter (dilation factor of 2), then move the cursor to point ***B***.
- Press enter to superimpose the dilation of triangle ***PBQ*** over triangle ***ABC***.

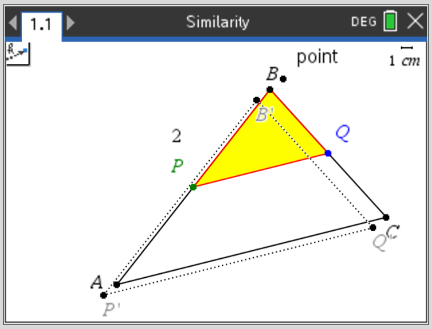


Answer. Triangle *P'B'Q'*, the image of triangle *PBQ* with dilation factor 2, is the same shape and size as triangle *ABC*.

... continued

Exploring similarity using Transformation tools in the Geometry application (continued)

(b) A consequence of the above is that the midsegment, ***PQ***, is parallel to side ***AC*** and half its length. This can be shown using conditions of similarity:

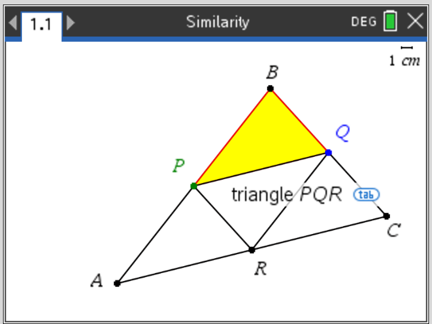


Answer. Since P, Q are midpoints, using the **SAS**[1] condition of similarity, triangles PBQ and ABC are similar. That is,

[1] **SAS**: two pairs of sides are in the same ratio ($\overline{AB} = 2\overline{PB}$, $\overline{BC} = 2\overline{BQ}$) and an equal included angle ($\measuredangle ABC = \measuredangle PBQ$).

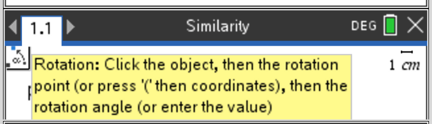
$\therefore d(PQ) = \frac{1}{2} d(AC)$ and $\overline{PQ} \parallel \overline{AC}$, as required.

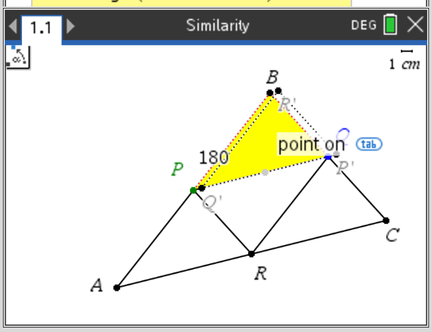
(c) To draw triangle PQR:



- Press menu > **Construction** > **Midpoint**. Click side ***AC***, then press esc to exit the tool.
- Label the midpoint of ***AC*** as ***R***.
- Press menu > **Shapes** > **Triangle**. Click on points ***P***, ***Q*** and ***R***, then press esc to exit the tool.

To visually verify congruence of the triangles ***PQR***, ***PBQ***, ***APR*** and ***RQC***, using the rotation transformation tool:





- Press menu > **Transformation** > **Rotation**.
- Click triangle ***PQR*** and press tab until $\triangle$***PQR*** is indicated.
- Press 1 8 0 enter (rotation of 180°), then move the cursor to superimpose the image of ***PQR*** (labelled ***P'Q'R'***) over each of the other three triangles.

Answer. In parts **(a)** and **(b)** above it was shown that triangle ABC is a dilation of triangle PBQ with a scale factor of 2. The visual verification that triangles PQR, PBQ, APR and RQC are congruent indicates that if the scale factor is 2, then the area of triangle ABC is 4 times that of triangle PBQ.

VCE General Mathematics Unit 3

3.1. Investigating data distributions

3.1.1. Displaying the distribution of a numerical variable

Using a logarithmic scale to display the distribution of a numerical variable

The estimated 2025 *population size* of the 11 countries in South-East Asia is shown in the table on the right.

(a) Construct a histogram of *population size* using an appropriate linear scale.

(b) Construct a histogram using the $\log_{10}$(*population size*).

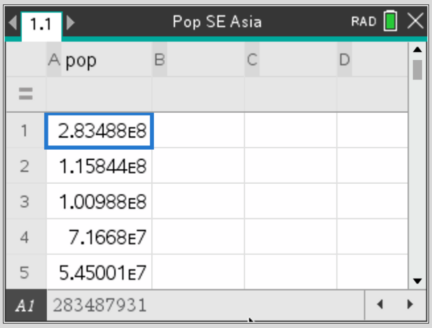
(c) Compare the two histograms constructed – which is more useful in displaying detail of the distribution of population size?

Country	*Population size*
Indonesia	283,487,931
Philippines	115,843,670
Vietnam	100,987,686
Thailand	71,668,011
Myanmar	54,500,091
Malaysia	35,557,673
Cambodia	17,638,801
Laos	7,769,819
Singapore	5,832,387
Timor-Leste	1,400,638
Brunei	462,721

On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable name ***pop***.
- Enter the data for ***pop*** into column A.

Note: *The data for population size may be displayed using scientific notation (see right). For instance, the screen shown right is from a calculator with display digits set to FLOAT 6. In this instance, values of population size greater than 6 digits will be displayed using scientific notation, with up to 6 digits in the mantissa. The display digits setting can be altered by pressing* [on] *>* ***Settings*** *>* ***Document Settings****.*

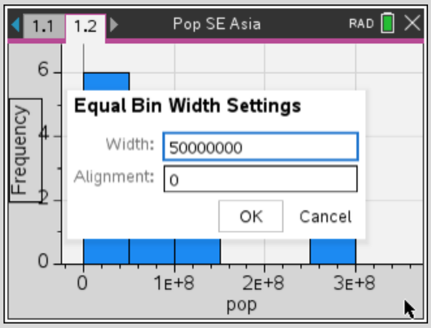


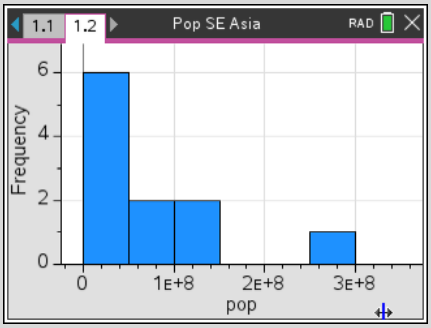
(a) Add a **Data & Statistics** page, and then:

- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select the variable ***pop***.

The default plot is a dot plot. Observing the data, a histogram column width of 50 million seems reasonable. To change the plot type to a histogram:

- Press [menu] > **Plot Type** > **Histogram**.
- Press [menu] > **Plot Properties** > **Histogram Properties** > **Bin Settings** > **Equal Bin Width**.
- Change **Width** to 50,000,000 (i.e. the width of each histogram column).
- Change **Alignment** to 0 (i.e. the starting value of ***pop*** for the histogram columns).
- Press [menu] > **Window/Zoom** > **Zoom – Data**.





Note: *The* ***Bin Width*** *of a histogram is the width of each column. The* ***Bin Alignment*** *of a histogram is the value from which the histogram columns should start.*

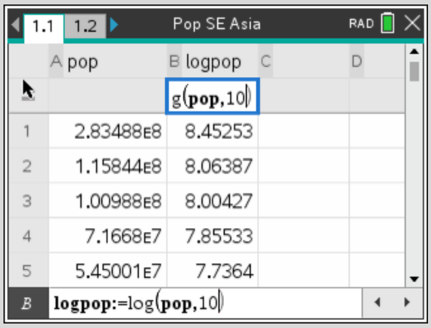
Moving the cursor over the histogram will confirm the frequencies for each ***pop*** interval of 50,000,000.

… continued

Using a logarithmic scale to display the distribution of a numerical variable (continued)

(b) To construct a histogram for the logarithm (using base 10) of *population size*, move back to the **Lists & Spreadsheet** page and then:

- In the column B heading cell, enter the name ***logpop***.
- In the column B formula cell, type **=log(*pop*,10)**.

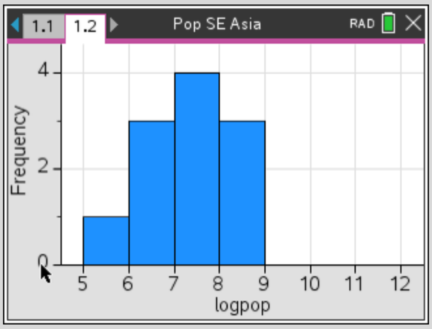


This will calculate and display the base 10 logarithmic numbers associated with each country's population size. For instance, Indonesia's population of 283,487,931 is approximately $10^{8.45253}$, and so the associated logarithm value is approximately 8.45253.

***Note:** If the **Calculation Mode** is set to **Auto** mode (via [on] > **Settings** > **Document Settings**), the calculator will display the values in the **logpop** column in exact form (e.g. $\log_{10}(283{,}487{,}931)$) rather than approximate decimal form (e.g. 8.45253). To display in approximate form, set the **Calculation Mode** to **Approx**, or include a decimal point in the formula (i.e. **=log(pop,10.0)**)*

To display the histogram for the logarithm of *population size*, move back to the **Data & Statistics** page and then:

- Click on the variable ***pop*** underneath the horizontal axis and select the variable ***logpop***.
- Press [menu] > **Plot Properties** > **Histogram Properties** > **Bin Settings** > **Equal Bin Width**.
- Change **Width** to 1 (i.e. the width of each histogram column).
- Change **Alignment** to 5 (i.e. the starting **logpop** value for the histogram columns).
- Press [menu] > **Window/Zoom** > **Zoom – Data**.



***Note:** The horizontal scale can be adjusted directly by clicking on the horizontal axis and then using the left or right arrows to adjust the scale.*

(c) Answer: The histogram using a linear display shows that the values of *population size* are spread out from just under 500,000 to just under 300,000,000, with 6 countries with *population size* between 0 and 50,000,000. It is difficult to view details of the entire distribution in this histogram.
In contrast to this, the histogram displaying the logarithmic scale brings the columns more closely together highlights that only one country has a population between 10^5 and 10^6 (i.e. between 100,000 and 1,000,000), whereas 3 countries have a population between 10^6 and 10^7 (i.e. between one and ten million). The histogram displaying the logarithmic scale makes it easier to display values on the horizontal axis, as they span between logarithmic values of 5 and 9.

3.1.2. Modelling with the normal distribution

Visualising the normal distribution

On *Able Farm*, the eggs produced are known to have a mean weight of 70 g, with a standard deviation of 10 g. Let A be the weight of eggs from *Able Farm* and assume that A can be modelled by a normal distribution.

(a) Create a plot of the distribution of A in a suitable viewing window.

(b) Shade the area under the graph associated with egg weights between 50 and 90 g.

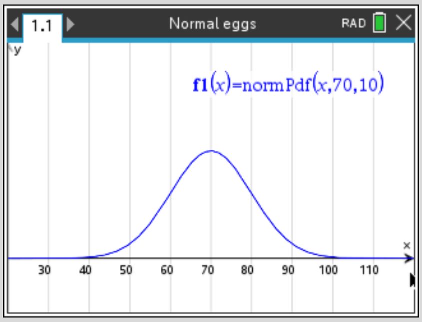
(c) Use the graph to find the approximate percentage of eggs for which

(i) $50 \le A \le 90$ **(ii)** $40 \le A \le 100$ **(iii)** $60 \le A \le 90$.

(a) To plot the distribution of A, on a **Graphs** page:

- Press [book] [N] and select **normPdf()**.
- Enter $f1(x) =$ **normPdf(*x*,70,10)**.
- Press [menu] > **Window/Zoom** > **Window Settings**

In the dialog box that follows, enter the following values:
XMin = 20 Xmax = 120 XScale = 10
YMin = −0.02 YMax = 0.08 YScale = 1

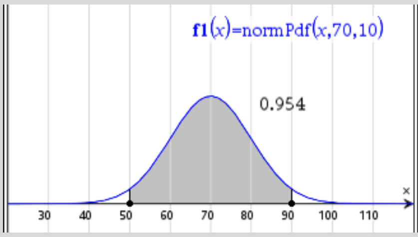


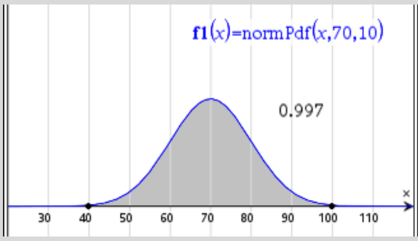
Note: *The screen shown right has been enhanced to display a lined grid (via* [menu] *>* ***View*** *>* ***Grid*** *>* ***Lined Grid****). The axis values are labelled, which can be displayed by hovering over one of the axes, then pressing* [ctrl] [menu] *>* ***Attributes****. In the popup menu, press ▼ to select the* ***Tick Labels*** *attribute, then press ◀ to select* ***Multiple Labels****. Press* [enter] *to save this attribute.*

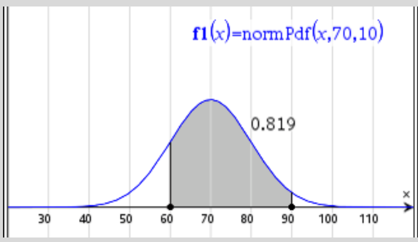
(b) To shade and calculate the area under the normal curve:

- Press [menu] > **Analyse Graph** > **Integral**.
- For 'lower bound', type **50** and then press [enter].
- For 'upper bound', type **90** and then press [enter].

The relevant area under the normal curve for part (c)(i) will be shaded, and an approximate value for the area is calculated and displayed. Press [ctrl] [Z] to remove the shading before each new area calculation.







(c) Answers:

(i) It is expected that approximately 95.4% of eggs will have weight between 50 and 90 g.

(ii) It is expected that the 99.7% of eggs will have a weight between 40 and 100 g.

(iii) It is expected that the 81.9% of eggs will have a weight between 60 and 90 g.

Note: *The percentages given in the answers here are* ***only*** *approximate values, as are the values referenced by the '68/95/99.7% rule', which are even more approximate! The example here is used mainly to help students visualise the approximate percentages based on the relevant area under the normal curve.*

... continued

Visualising the normal distribution (continued)

On *Bonza Farm*, the eggs produced are known to have a mean weight of 70 g, with a standard deviation of 5 g. Let B be the weight of eggs from *Bonza Farm*, and assume that B can also be modelled by a normal distribution.

(d) Add a plot of the distribution of B underneath the plot of the distribution of A.
(e) Use the two plots to help explain why an egg weight of greater than 80 g might be more common on *Able Farm* than *Bonza Farm*.

(d) To add a plot of the distribution of B, on the **Graphs** page:

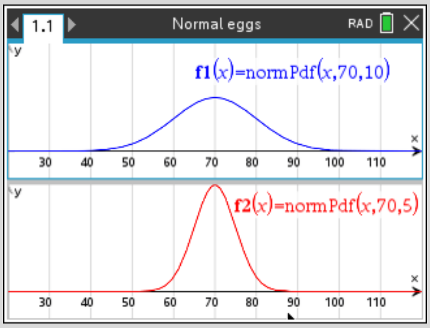
- Press [doc▾] > **Page Layout** > **Select Layout** > **Layout 3**
- Click in the bottom half of the screen and add another **Graphs** page.
- Enter $f2(x) =$ **normPdf(x,70,5)**.
- Press [menu] > **Window/Zoom** > **Window Settings**

In the dialog box that follows, enter the following values:

XMin = 20	Xmax = 120	XScale = 10
YMin = −0.02	YMax = 0.08	YScale = 1

- Press [menu] > **View** > **Grid** > **Lined Grid**.
- Hover over the horizontal axis, then press [ctrl] [menu] > **Attributes**. In the popup menu, press ▼ to select the **Tick Labels** attribute, then press ◀ to select **Multiple Labels**. Press [enter] to save this attribute.

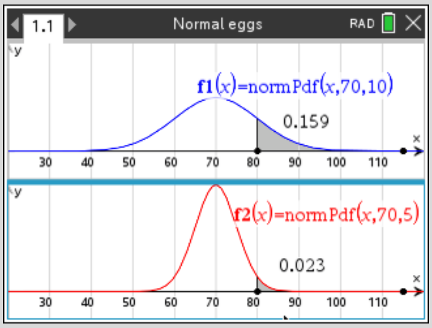
This displays the distribution of A and B on the same scale.



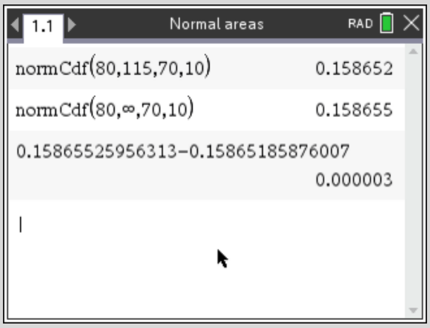
Note: It is not a trvial skill for students to find stuitable boundaries for the viewing window. Discussing the value of the mean and standard deviation will help, as is the need to adjust the vertical scale. ***Zoom-Fit*** *is helpful here.*

(e) To shade and calculate the area under each of the normal curves:

- Click in the top half of the screen (*Able Farm* curve)
- Press [menu] > **Analyse Graph** > **Integral**.
- For 'lower bound', type **80** and then press [enter].
- For 'upper bound', type **115** and then press [enter].*
- Repeat the above steps for the lower half of the screen (for *Bonza Farm* curve)



Answer: Comparing the areas, approximately 16% of the eggs on *Able Farm* will have weights greater than 80 g, whereas approximately 2% of the eggs on *Bonza Farm* will have weights greater than 80 g. Therefore, an egg weight of greater than 80 g will be more common on *Able Farm* than *Bonza Farm.* An 80 g egg from *Able Farm* represents a weight only one standard deviation above the mean weight, whereas an 80 g egg from *Bonza Farm* represents a weight two standard deviations above the mean weight.



****Note:** To estimate an area associated with the statements 'greater than 80 g', it is reasonable to select an upper bound which is sufficiently large (e.g. 115 g). The screen right highlights the negligible difference in the area for a chosen upper bound of 115 g rather than the theoretical upper bound of 'infinity' grams!*

Solving with z-scores and a 'wildcard' symbol

Test scores on the 'Weschler Adult Intelligence Scale' (often used in so-called 'IQ tests') are normally distributed with mean 100 marks and standard deviation 15 marks.

(a) What z–score corresponds to a test score of 130?

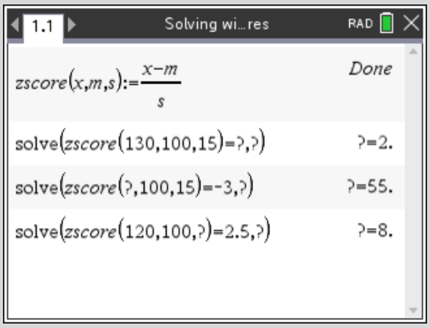
(b) What test score corresponds to a z–score of –3?

(c) In a modified version of the above test, test scores are normally distributed with mean 100, but the standard deviation is different. If a test score of 120 now corresponds to a z–score of 2.5, what is the standard deviation for the modified test?

The formula for standard normal score (z–score) is $z = \dfrac{x - \bar{x}}{s_x}$.

To construct a calculator formula for the z–score, on a **Calculator** page:

- Enter $\boldsymbol{zscore(x, m, s) := \dfrac{x - m}{s}}$.



(a) To find the z–score for a test score of 130:

- Enter $\boldsymbol{solve(zscore(130,100,15) = ?, ?)}$.

(b) To find the *test score* for a z–score of –3:

- Enter $\boldsymbol{solve(zscore(?,100,15) = -3, ?)}$.

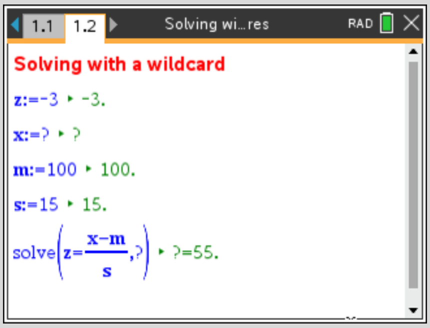
(c) To find the new standard deviation for a test score of 120:

- Enter $\boldsymbol{solve(zscore(120,100,?) = 2.5, ?)}$.

Answer: (a) $z = 2$ **(b)** Test score = 55 **(c)** $s_x = 8$

Note: *The '?' symbol can be found by pressing* [?!▸]. *It is referred to here as a 'wildcard' symbol, as it can be used to represent whichever one of the variables has an unknown value. In this way the standard formula can be used, substituting the wildcard symbol for the unknown variable in the solve command.*

To create a **Notes** page utilising the same approach, add a **Notes** page and then (using the example from part **(b)** above):



- Enter the title text as shown.
- Press [ctrl] [M] to insert a Maths Box and then:
  - Enter $\boldsymbol{z := -3}$
  - Enter $\boldsymbol{x := ?}$
  - Enter $\boldsymbol{m := 100}$
  - Enter $\boldsymbol{s := 15}$
  - Enter $\boldsymbol{solve(z = \dfrac{x - m}{s}, ?)}$.

To solve for a particular z–score, test score, mean or standard deviation, enter the known values, and then enter '?' for the unknown value. The solution will be visible in the last line.

Note: *This method of using the wildcard symbol (?) can be used for solving most formulas used in the General Mathematics course.*

3.2. Investigating association between two variables

3.2.1. Correlation

Finding the correlation coefficient

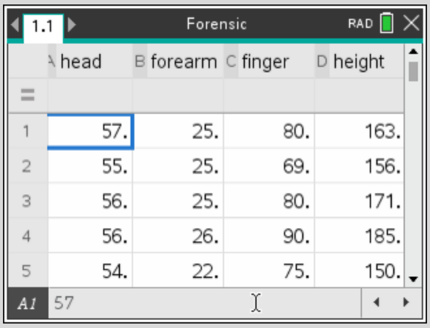
Four body measurements were taken from 15 students. The following table displays this data.

Head circumference (cm)	57	55	56	56	54	58	57	57	59	59	57	61	58	59	57
Forearm length (cm)	25	25	25	26	22	27	25	27	26	23	24	28	27	23	24
Middle finger length (mm)	80	69	80	90	75	89	80	82	79	80	78	85	95	74	85
Height (cm)	163	156	171	185	150	169	150	172	175	169	166	188	179	162	170

(a) Find the correlation between each pair of variables (using Pearson's correlation coefficient *r*).
(b) Which pair of variables have the highest correlation?

(a) To enter the above data, on a **Lists & Spreadsheet** page:

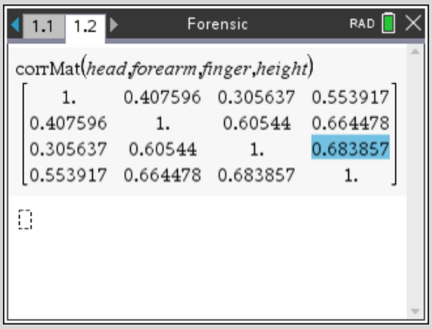
- In the column A heading cell, enter the name ***head***.
- In the column B heading cell, enter the name ***forearm***.
- In the column C heading cell, enter the name ***finger***.
- In the column D heading cell, enter the name ***height***.
- Enter the data for the four variables into columns A to D.



To calculate the values of *r*, add a **Calculator** page and then:

- Press menu > **Statistics** > **Stat Calculations** > **Correlation Matrix.**
- Press var and select each variable in turn, separated by the comma symbol (as shown right).

This will display a matrix of the values of *r* for each pair of variables (in the order that the variables were entered).

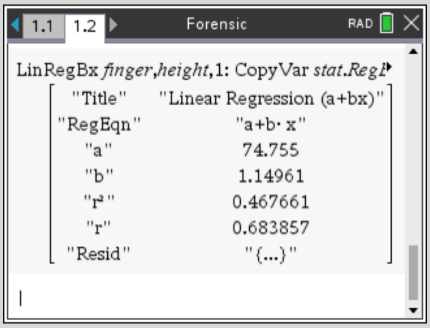


Note: *For 4 variables, there will be 6 variable pairings. The main diagonal values of r reflect perfect correlations, as they are pairings of the same variable. There are repeated values due to the symmetry properties of a square matrix (e.g. if X is the correlation matrix, then* $X_{1,2} = X_{2,1}$*).*

(b) From the correlation matrix screen shown right, the highlighted value of $r = 0.683857$ is in a cell located in column 4 and row 3, and so this value represents the correlation between the variables with the highest correlation, *height* and *finger*.

The value of the correlation coefficient for a pair of numerical variables can also be found by calculating the least squares regression line. For instance, for the two variables *height* (response variable) and *finger* (explanatory variable):

- Press menu > **Statistics** > **Stat Calculations** > **Linear Regression (a+bx)**.
- For 'X List', select ***finger***.
- For 'Y List', select ***height***.



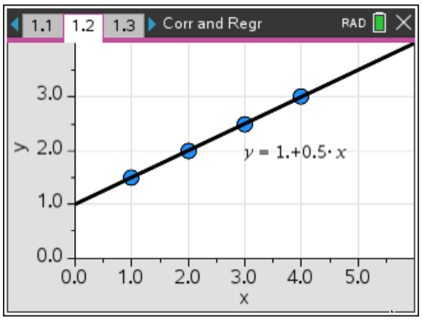
This will display both the equation of the least squares regression line and the values of *r* and r^2.

3.2.2. Least squares regression

Linking correlation and least-squares regression

Consider the following dataset, where the regression line (using form $y = a + bx$) with equation $y = 1 + 0.5x$ is a perfect linear model (that is, $r = 1$).

x	1	2	3	4
y	1.5	2	2.5	3



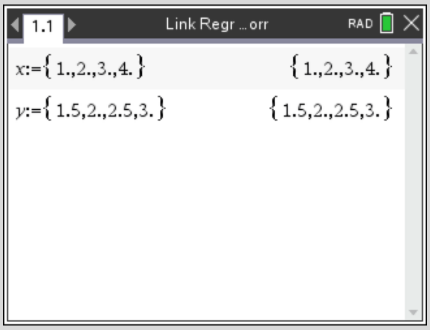
The scatter plot and regression line are shown right, highlighting that the line passes perfectly through the four points.

(a) Calculate the values of s_x and s_y, then calculate the ratio $\frac{s_y}{s_x}$. What do you notice?

(b) Calculate the value of $\bar{x}$ and $\bar{y}$. then calculate the value of $\bar{y} - 0.5\,\bar{x}$. What do you notice?

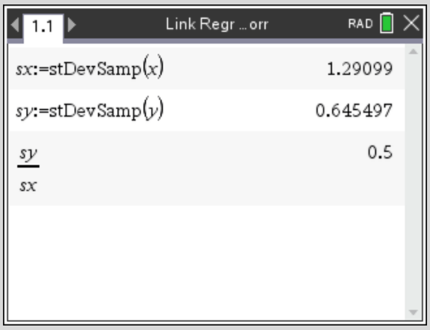
To enter the data, on a **Calculator** page:

- Type the variable name ***x***.
- Press ctrl ▫{⁝ to enter the 'Assign' symbol.
- Press ctrl) to enter the braces (set brackets).
- Enter the ***x*** data as shown.
- Repeat the above steps for the variable ***y***.



(a) To calculate the standard deviation of ***x*** and ***y***:

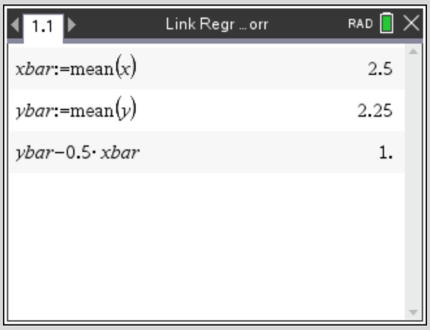
- Type the variable name ***sx***.
- Press ctrl ▫{⁝ to enter the 'Assign' symbol.
- Press menu **> Statistics > List Maths > Sample Standard Deviation.**
- Press var and select the variable ***x***.
- Repeat these steps for the variable ***sy***.
- Enter ***sy***/***sx*** to calculate the value of the ratio $\frac{s_y}{s_x}$



Answer: The value of the ratio $\frac{s_y}{s_x} = 0.5$. This is the same as the value of the gradient b of the regression line.

(b) To calculate the mean of ***x*** and ***y***:

- Type the variable name ***xbar***.
- Press ctrl ▫{⁝ to enter the 'Assign' symbol.
- Press menu **> Statistics > List Maths > Mean.**
- Press var and select the variable ***x***.
- Repeat these steps for the variable ***ybar***.



To calculate the value if $\bar{y} - 0.5\,\bar{x}$:

- Enter ***ybar – 0.5xbar***.

Answer: The value of $\bar{y} - 0.5\,\bar{x} = 1$. This is the same as the value of the y-intercept a of the regression line.

... continued

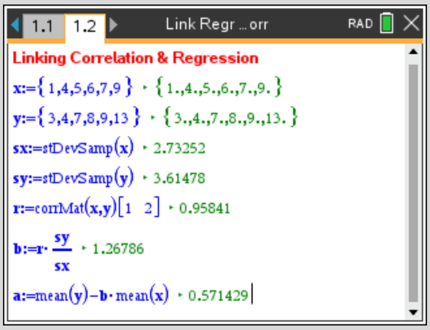
Linking correlation and least-squares regression (continued)

***Note**: The following comments and screens are included to illustrate how selected summary statistics can be used to show the link between correlation and the associated least squares regression equation. It may be of interest to show students this link.*

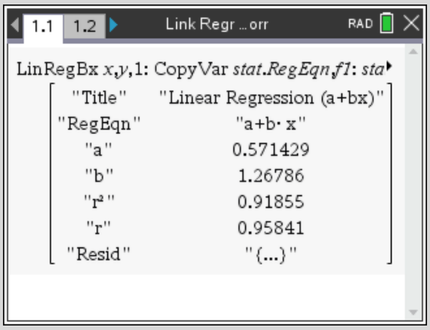
For the preceding 'perfect' correlation example, the values of least-squares regression parameters for the slope (b) and y-intercept (a) were linked, with $b = \dfrac{s_y}{s_x}$ and $a = \overline{y} - b\overline{x}$.

In the more common scenario where the correlation between the two numeric variables is not perfect, the following linking formulae can be applied.

$$b = r\frac{s_y}{s_x} \text{ and } a = \overline{y} - b\overline{x}$$



In the screen top right, a **Notes** page has been constructed to calculate the values of the slope (b) and y-intercept (a) based on the mean and standard deviation values of x and y, along with the value of the correlation coefficient r.



For comparison, the screen right, shows the same results on a **Calculator** page, obtaining by pressing menu **> Statistics > Stat Calculations > Linear Regression (a + bx).** This shows that the same values for the slope (b) and y-intercept (a) are obtained via the least-squares regression algorithm directly.

Finding the least-square regression line using the List & Spreadsheet App

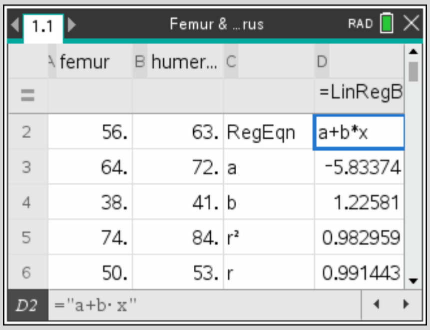
Length measurements of the femur bone (in the leg) and humerus (upper arm) were made on fossils of a particular species. A researcher is interested in whether the length of the femur is a good predictor of the length of the humerus (so the length of the femur is the explanatory variable).

Length of femur (cm)	59	56	64	38	74	50
Length of humerus (cm)	70	63	72	41	84	53

Find the least squares regression equation which can be used to predict humerus length from femur length. Then use this to predict the length of the humerus for a fossil whose femur has a known length of 80 cm.

To enter the above data, on a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the name ***femur***.
- In the column B heading cell, enter the name ***humerus***.
- Enter the data for the four variables into columns A to B.



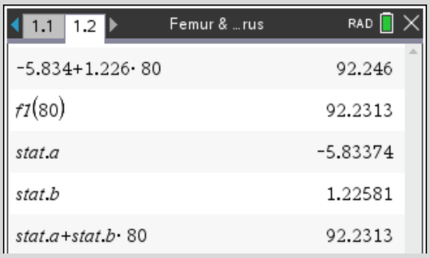
To find the equation of the least squares regression line:

- Press [menu] **> Statistics > Stat Calculations > Linear Regression (a + bx).**
- For 'X List', click and select the variable **femur**.
- For 'Y List', click and select the variable **humerus**.
- Click **Ok** to display the results in columns C and D

Answer: The equation of the least squares regression line is approximately (to 4 significant figures):

$$length\ of\ \ humerus = -5.834 + 1.226 \times length\ of\ \ femur\,.$$

Using a **Calculator** page, if the length of the femur is 80 cm, the predicted length of the humerus is:



$$length\ of\ \ humerus = -5.834 + 1.226 \times (80) \approx 92 \text{ cm}.$$

***Note**: The least squares regression equation by default will be stored in the function variable **f1**. On a **Calculator** page, entering **f1(80)** will calculate the predicted humerus length for a femur length of 80 cm. The answer obtained will be more accurate than the answer obtained using rounded values for **a** and **b**. Note also that you can access the values of **a** and **b** via the [var] key (see screen right). The variables **stat.a** and **stat.b** contain the values of the y-intercept and slope respectively obtained from the most recent statistical calculation.*

The importance of visualising for understanding

Here are four famous datasets (*Anscombe's Quartet*) that highlight the need to visualise the association between two numerical variables, rather than solely relying on summary statistics.

Dataset 1

$x1$	10	8	13	9	11	14	6	4	12	7	5	
$y1$	8.04	6.95	7.58	8.81	8.33	9.96	7.24	4.26	10.84	4.82	5.68	

Dataset 2

$x2$	10	8	13	9	11	14	6	4	12	7	5	
$y2$	9.14	8.14	8.74	8.77	9.26	8.10	6.13	3.10	9.13	7.26	4.74	

Dataset 3

$x3$	10	8	13	9	11	14	6	4	12	7	5	
$y3$	7.46	6.77	12.74	7.11	7.81	8.84	6.08	5.39	8.15	6.42	5.73	

Dataset 4

$x4$	8	8	8	8	8	8	8	19	8	8	8	
$y4$	6.58	5.76	7.71	8.84	8.47	7.04	5.25	12.50	5.56	7.91	6.89	

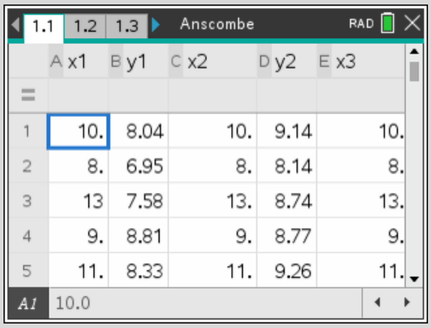
(a) For each of the datasets shown, find:

- the mean and standard deviation of each of the variables in the dataset
- the correlation coefficient and the equation of the least squares regression line

(b) Construct a scatter plot for each data set and compare these plots.

(a) To enter the above data, on a **Lists & Spreadsheet** page:

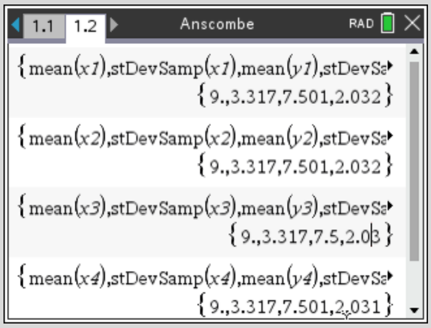
- In the column heading cells A to H, enter the variable names ***x*1**, ***y*1**, ***x*2**, ***y*2**, ***x*3**, ***y*3**, ***x*4**, ***y*4**.
- Enter the data for the eight variables into columns A to H.



To calculate the mean and standard deviation of each variable, add a **Calculator** page and then:

- Press [ctrl] [)] to enter a set of braces (set brackets).
- Enter the command $\{\textbf{mean}(x1), \textbf{stDevSamp}(x1), \textbf{mean}(y1), \textbf{stDevSamp}(y1)\}$.

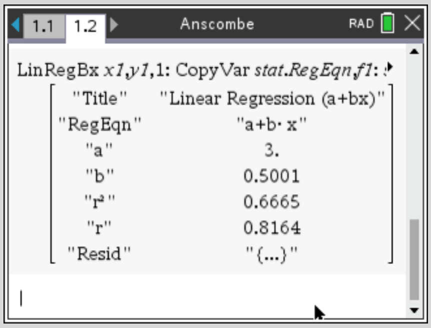
Note: The mean and sample standard deviation commands can be found via [menu] ***> Statistics > List Maths***.



Answer: The means of the x-variables have the same mean and standard deviation $(\bar{x} = 9, s_x \approx 3.32)$, and the y-variables have approximately the same mean and standard deviation $(\bar{y} = 7.5, s_y \approx 2.03)$.

To calculate the correlation coefficients and the equation of the least squares regression lines:

- Press [menu] **> Statistics > Stat Calculations > Linear Regression (a+bx)**.
- For X list, select ***x1***.
- For Y List, select ***y1***.
- Repeat this for each dataset.



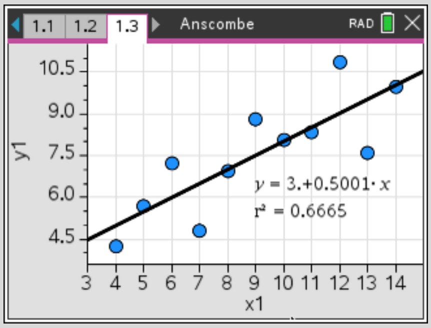
Answer: The values of the correlation coefficient $(r \approx 0.816)$ and least squares regression equation ($y = 3 + 0.5x$) for each dataset are approximately equal.

... continued

The importance of visualising for understanding (continued)

(b) To display the scatter plot for Dataset 1, add a **Data & Statistics** page, and then:

- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select the variable ***x1***.
- Press [tab] to activate **Click to add variable** to the left of the vertical axis and select the variable ***y1***.
- To show the least squares regression line, press [menu] > **Analyse** > **Regression** > **Show Linear (a + bx)**.

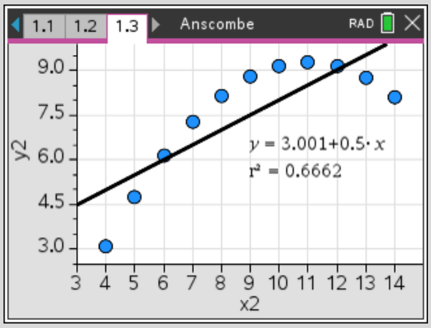


This will display a scatter plot for Dataset 1, with the least squares line drawn over the plot.

Note: *Click on the line to display the equation of the least squares regression line and the coefficient of determination. Press [menu] > **Settings** and adjust the **Display Digits** as required. The **Diagnostics** tick box is to show/hide the coefficient of determination.*

To display the scatter plot for Dataset 2:

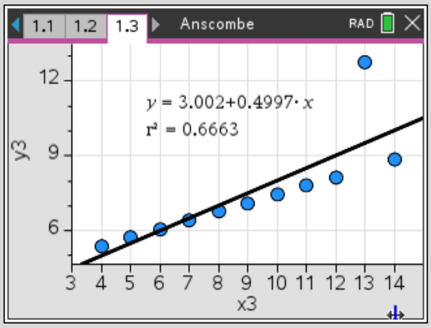
- Click the variable underneath the horizontal axis and select the variable ***x2***.
- Click the variable to the left of the vertical axis and select the variable ***y2***.



This scatter plot reveals a clear non-linear pattern not obvious from the summary statistics calculated.

To display the scatter plot for Dataset 3:

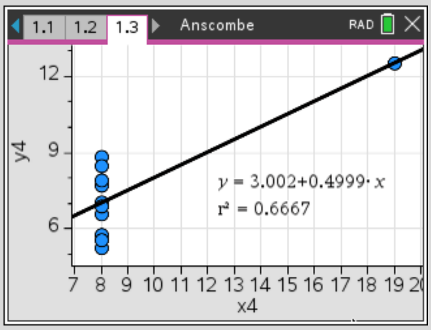
- Click the variable underneath the horizontal axis and select the variable ***x3***.
- Click the variable to the left of the vertical axis and select the variable ***y3***.



This scatter plot reveals the impact of an outlier point on the summary statistics for an otherwise perfect linear pattern.

To display the scatter plot for Dataset 4:

- Click the variable underneath the horizontal axis and select the variable ***x4***.
- Click the variable to the left of the vertical axis and select the variable ***y4***.



This scatter plot reveals clustering of all but one point around a set of points with the same value of the explanatory variable.

3.3. Investigating and modelling linear associations

3.3.1. Analysing linear models

Constructing a template file for analysing linear association

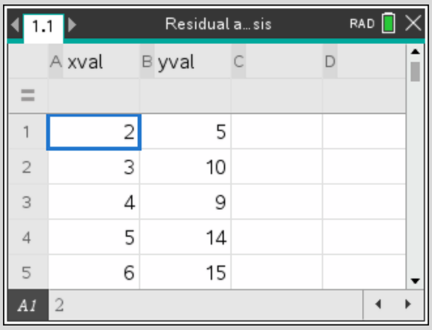
Consider the following bivariate dataset. Assume that x is the explanatory variable.

x	2	3	4	5	6
y	5	10	9	14	15

(a) Construct a scatter plot for each data set and compare these plots.
(b) Find the equation of the least squares regression line and the coefficient of determination
(c) Display a plot of the residual values, and comment on whether a linear model is reasonable.

(a) To enter the above data, on a **Lists & Spreadsheet** page:

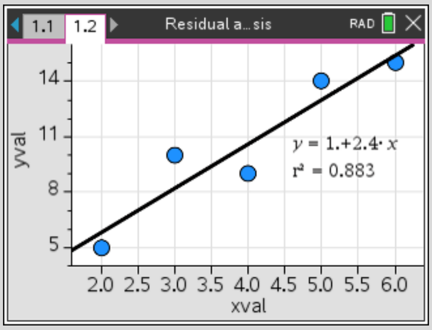
- In the column heading cells A and B, enter the variable names ***xval*** and ***yval***.
- Enter the data for the two variables into columns A and B.



***Note**: If column(s) A and B are not empty (that is, there is data in column under **xval** and/or **yval**), press the ▲ key until the entire column is selected, and then press* [menu] *> **Data** > **Clear Data**. This ensures that the previous data is removed.*

(b) To display the scatter plot and add a least squares regression line, add a **Data & Statistics** page, and then:

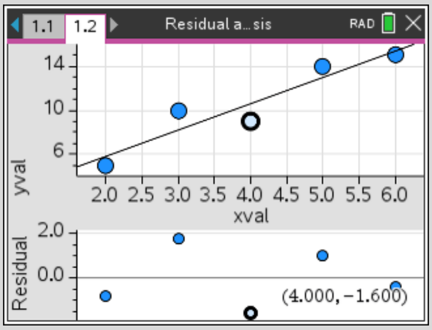
- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select the variable ***xval***.
- Press [tab] to activate **Click to add variable** to the left of the vertical axis and select the variable ***yval***.
- To show the least squares regression line, press [menu] > **Analyse** > **Regression** > **Show Linear (a + bx)**



This will display a scatter plot, with the least squares line drawn over the plot. Click on the line to display the equation of the least squares regression line and the coefficient of determination.

***Note**: If the coefficient of determination is not displayed, press* [menu] *> **Settings** and click the **Diagnostics** option.*

(c) To display a residual plot, press [menu] > **Analyse** > **Residuals** > **Show Residual Plot.**



***Note**: Clicking a point on the residual plot highlights the associated point on the scatter plot, and displays the residual plot. For example, in the screen shown right, the point at (4,9) on the scatter plot has an associated residual value of −1.6, meaning that the actual y-value is 1.6 units below the predicted value for x = 4.*

Answer: The residuals appear to be small and randomly placed about zero, and so a linear model is appropriate.

3.3.2. Analysing non-linear models

Transforming variables to identify a better model

A stone is dropped from a bridge that is 50 metres above the water, and a photo is taken (from the side) every 0.5 seconds, yielding the data shown about the height of the stone above the water over the first three seconds.

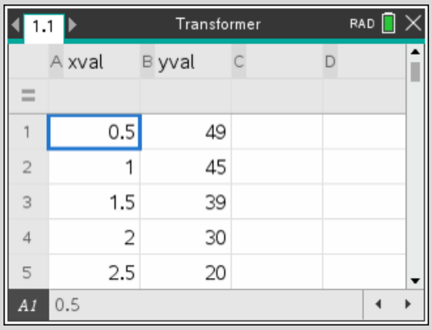
Time elapsed (*s*)	0.5	1.0	1.5	2.0	2.5	3.0
Height of stone (*m*)	49	45	39	30	20	6

(a) Construct a scatterplot and fit a regression line to this data. Plot the residuals and use this to comment on the suitability of the linear regression model.
(b) Construct formulas to transform the explanatory and response variables so that non-linear regression models can be checked. Include squared, logarithmic and reciprocal transformations of both the explanatory and response variables.
(c) It has been suggested that a non-linear model $height = a + b \times time^2$ may fit the model well. Check this model and comment on its suitability.

(a) To enter the above data, on a **Lists & Spreadsheet** page:

- In the column heading cells A and B, enter the variable names ***xval*** and ***yval***.
- Enter the data for the two variables into columns A and B.

Note: *If column(s) A and B are not empty (that is, there is data in column under* ***xval*** *and/or* ***yval****), press the ▲ key until the entire column is selected, and then press* menu *>* ***Data*** *>* ***Clear Data****. This ensures that the previous data is removed.*

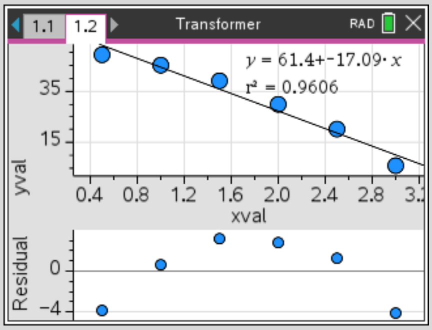


To display the scatter plot and add a least squares regression line, add a **Data & Statistics** page, and then:

- Press tab to activate **Click to add variable** underneath the horizontal axis and select the variable ***xval***.
- Press tab to activate **Click to add variable** to the left of the vertical axis and select the variable ***yval***.

To show the least squares regression line and display the residual plot:

- Press menu > **Analyse** > **Regression** > **Show Linear (a+bx)**.
- Press menu > **Analyse** > **Residuals** > **Show Residual Plot**.



This will display a scatter plot, with the least squares line drawn over the plot. Click on the line to display the equation of the least squares regression line and the coefficient of determination.

Note: *If the coefficient of determination is not displayed, press* menu *>* ***Settings*** *and click the* ***Diagnostics*** *option.*

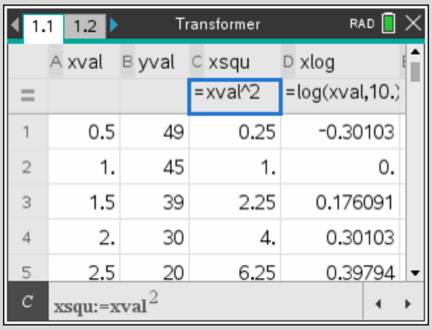
Answer: Despite the high value of $r^2 = 0.96$, the residuals form a curved pattern, suggesting that a non-linear model may better fit the data.

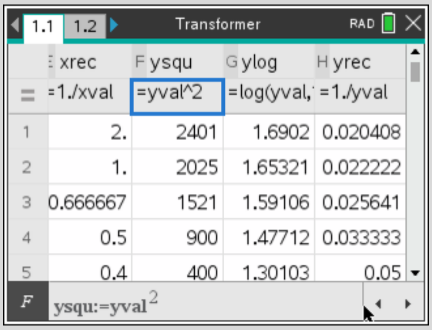
... continued

Transforming variables to identify a better model (continued)

(b) Press [ctrl] ◀ to return to the **Lists & Spreadsheet** page, and then:

- Add the variable names for the transformed variables:
  - In the heading cell for column C, enter the name ***xsqu***
  - In the heading cell for column D, enter the name ***xlog***
  - In the heading cell for column E, enter the name ***xrec***
  - In the heading cell for column F, enter the name ***ysqu***
  - In the heading cell for column G, enter the name ***ylog***
  - In the heading cell for column H, enter the name ***yrec***.
- Add the column formulas for each variable as shown:
  - In the formula cell for column C, enter the formula ***xsqu:=xval²***
  - In the formula cell for column D, enter the formula ***xlog:=log(xval,10.0)***
  - In the formula cell for column E, enter the formula ***xrec:=1.0/xval***
  - In the formula cell for column F, enter the formula ***ysqu:=yval²***
  - In the formula cell for column G, enter the formula ***ylog:=log(yval,10.0)***
  - In the formula cell for column H, enter the formula ***yrec:=1.0/yval***.

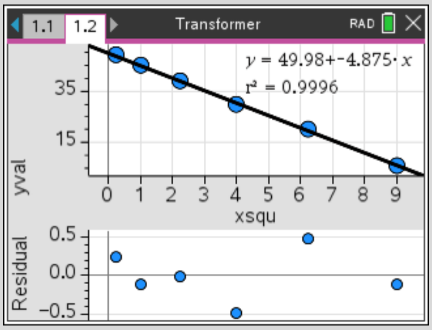




***Note**: The use of decimal point in some of the formulas is just to force the data to be displayed as decimal approximations. If the calculator setting is in APPROX calculation mode, then this is not necessary.*

(c) Press [ctrl] ▶ to return to the **Data & Statistics** page, and then:

- Click on the variable label ***xval*** (the variable on the horizontal axis).
- In the pop-up box that follows, select the variable ***xsqu***.
- Press [menu] **> Window/Zoom > Zoom – Data**.



Answer: The plot of height values against the (time)2 values linearises the plot very well, and the coefficient of determination is very high ($r^2 = 0.9996$). The residuals seem small and randomly distributed around zero, so the transformed model *height* = *49.98 – 4.875* × *time*^{2} seems to fit the data well.

***Note**: The number of display digits given for the regression equation and the coefficient of determination can be changed by pressing [menu] **> Settings** and modifying the number of display digits.*

3.4. Investigating and modelling time series data

3.4.1. Smoothing time series data

Constructing a spreadsheet to create a smoothing template

A spreadsheet is ideal for calculating smoothed values of a time series, with each column corresponding to a particular type and degree of smoothing. Here a spreadsheet template for these calculations will be constructed, suitable for up to 20 time periods. For the purposes of the template, the following time series will be used.

The number of births in a remote country hospital is shown below

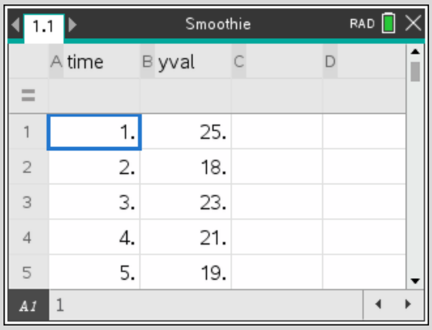
Year	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025
Number of births	25	18	23	21	19	20	18	16	17	15

Investigate smoothing the data with moving mean and median averages.

To enter the above time series data, on a **Lists & Spreadsheet** page:

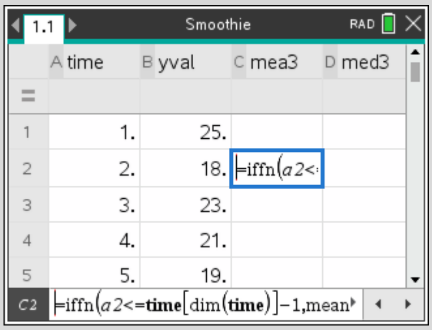
- In the column heading cells A and B, enter the variable names ***time*** and ***yval***.
- In column A, enter the data for ***time*** periods using the numbers 1 to 10 (i.e. let 2016 = 1, 2017 = 2 and so on).
- In column B, enter the data for the production (as ***yval***)

__Note__: If column(s) A and B are not empty (that is, there is data in column under __time__ and/or __yval__), press the ▲ key until the entire column is selected, and then press menu *>* ***Data > Clear Data****. This ensures that the previous data is removed.*



To add the variable names for the smoothed variables:

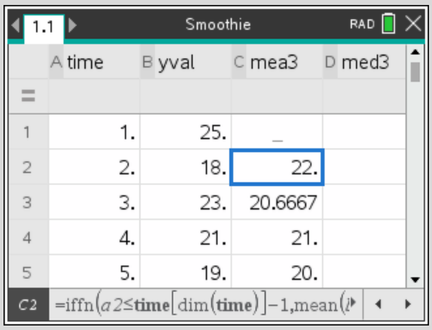
- In the heading cell for column C, enter the name ***mea3***
- In the heading cell for column D, enter the name ***med3***
- In the heading cell for column E, enter the name ***mea5***
- In the heading cell for column F, enter the name ***med5***
- In the heading cell for column G, enter the name ***mea4***
- In the heading cell for column H, enter the name ***cmea4***



To add the smoothing formulas for each column:

- For ***mea3***:
  - Click in C2, and enter the formula $= \textbf{iffn}\left(\textbf{a2} \le \textbf{\textit{time}}[\textbf{dim}(\textbf{\textit{time}})] - \textbf{1}, \textbf{mean}(\textbf{b1}:\textbf{b3}), \text{" "}\right)$
  - Click in cell C2, and press menu > Data > Fill
  - Press ▼ to select from C2 to C20 and press enter.

This will fill down the three-point moving mean value formula to the cells in this column down to C20, leaving a blank cell when required.

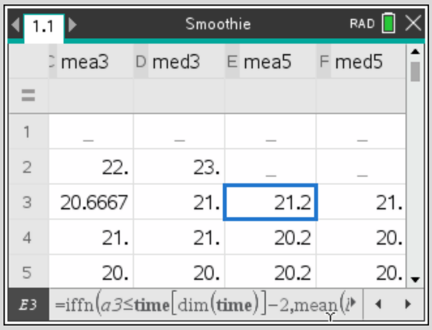


__Note__: In the template, the number of time periods has been set to 20, to ensure that it works for a range of examples.

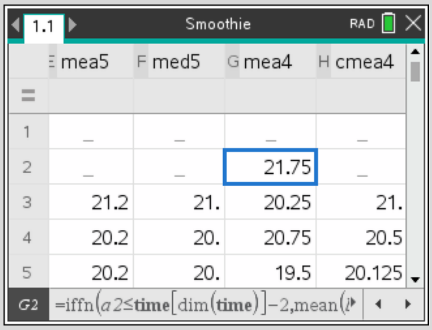
...continued

Constructing a spreadsheet to create a smoothing template (continued)

- For ***med3***:
  - Click in D2, and enter the formula
    $$=\text{iffn}\left(\text{a2} \le \textit{time}[\dim(\textit{time})]-1, \text{median}(\text{b1}:\text{b3}), ""\right)$$
  - Click in cell D2, and press menu > Data > Fill
  - Press ▼ to select from D2 to D20 and press enter.
- For ***mea5***:
  - Click in E3, and enter the formula
    $$=\text{iffn}\left(\text{a3} \le \textit{time}[\dim(\textit{time})]-2, \text{mean}(\text{b1}:\text{b5}), ""\right)$$
  - Click in cell E3, and press menu **> Data > Fill**
  - Press ▼ to select from E3 to E20 and press enter.
- For ***med5***:
  - Click in F3, and enter the formula
    $$=\text{iffn}\left(\text{a3} \le \textit{time}[\dim(\textit{time})]-2, \text{median}(\text{b1}:\text{b5}), ""\right)$$
  - Click in cell F3, and press menu **> Data > Fill**
  - Press ▼ to select from F3 to F20 and press enter.



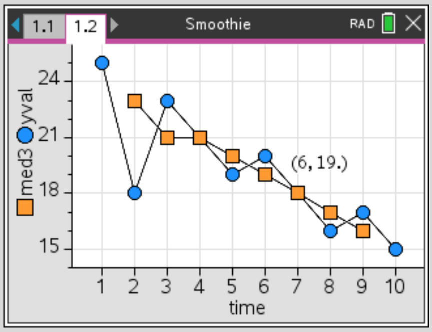
- For ***mea4***:
  - Click in G2, and enter the formula
    $$=\text{iffn}\left(\text{a2} \le \textit{time}[\dim(\textit{time})]-2, \text{mean}(\text{b1}:\text{b4}), ""\right)$$
  - Click in cell G2, and press menu **> Data > Fill**
  - Press ▼ to select from G2 to G20 and press enter.
- For ***cmea4***:
  - Click in H3, and enter the formula
    $$=\text{iffn}\left(\text{a3} \le \textit{time}[\dim(\textit{time})]-2, \text{mean}(g2:g3), ""\right)$$
  - Click in cell H3, and press menu **> Data > Fill**
  - Press ▼ to select from H3 to H20 and press enter.



***Note**: The values for **mea4** are not lined up horizontally with **time**, but the values of **cmea4** are lined up correctly.*

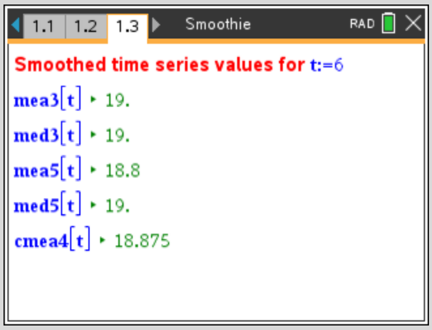
To view a plot of the time series along with a smoothed plot (for example the 3-point smoothed values), add a **Data & Statistics** page and then:

- Press tab and select ***time*** as the explanatory variable
- Press tab and select ***yval*** as the response variable
- Press menu **> Plot Properties > Connect Data Points**
- Press menu **> Plot Properties > Add Y Variable** and select the variable ***med3***.



The smoothed values for a specific time period can be displayed by hovering over the relevant point.

***Note**: The smoothed values for a specific time period can also be displayed by constructing a **Notes** page. A possible example is shown in the screen right.*



3.4.2. Analysing time series with seasonalised data

Constructing a spreadsheet to deseasonalise a time series

The following data is a record of quarterly sales for Yarra Bookstore. The table below gives the number of magazines sold in each of the four seasons for the years 2022 to 2024.

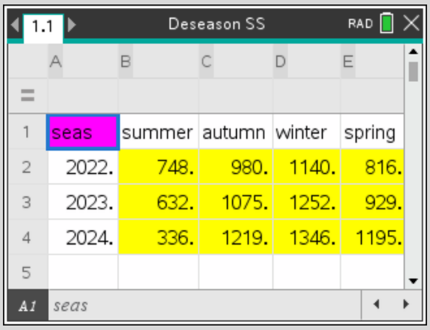
Year \ **Season**	*Summer*	*Autumn*	*Winter*	*Spring*
2022	748	980	1140	816
2023	632	1075	1252	929
2024	336	1219	1346	1195

Deseasonalise these sales figures and use a trend line to estimate the sales for Autumn 2025.

To enter the data, on a **Lists & Spreadsheet** page:

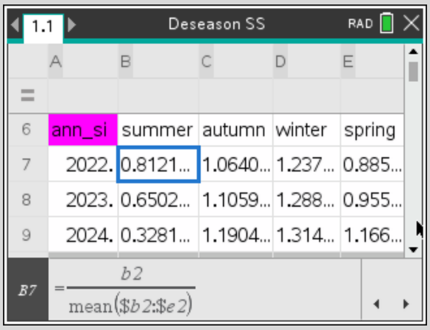
- In A1, enter the title 'seas', and then in A2 to A4, enter the years as shown.
- In B1 to E1, enter the titles of the seasons.
- In B2 to E4, enter the quarterly sales figures as shown.

Note**: To change the filled colours for a selected cell, click in the relevant cell, or press* [shift] *and the arrow keys to select a range of cells. Then click* [ctrl] [menu] *>* ***Color *>* ***Fill Color*** *and select the desired fill colour.*



To calculate the quarterly seasonal index (for each year):

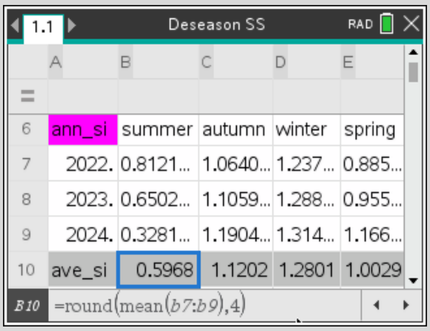
- In A6, enter the title 'ann_si', and then in A7 to A9, enter the years as shown.
- In B6 to E6, enter the titles of the seasons.
- In B7, enter the formula **=b2/mean($b2:$e2)**.
- Click in B7, and press [menu] > **Data** > **Fill**, and then press ▼ to select B7:B9.
- Press [enter] to fill the formula in B7 into B8:B9.
- Click in B7, then press [shift] ▼ to select B7:B9
- Press [menu] > **Data** > **Fill**, and press ▶ to select B7:E9.
- Press [enter] to fill the formulae from B7:B9 into C7:E9.



To calculate the average quarterly seasonal index:

- In A10, enter the title 'ave_si'.
- In B10, enter the formula **=round(mean(b7:b9),4)**
- Click in B10, then press [menu] > **Data** > **Fill**, and press ▶ to select B10:E10.
- Press [enter] to fill the formula from B10 into cells C10:E10.

The average seasonal indices are now displayed in row 10 of the spreadsheet, correct to four decimal places.



***Note**: The '$' symbol within a formula is used to make a cell reference 'absolute' rather than 'relative'. This is very useful when copying formulas (or 'filling'), as it allows to user to specify whether row/column references are fixed (will not change when copied to other cells), and which are relative to a row/column (will change when copied to other cells).*

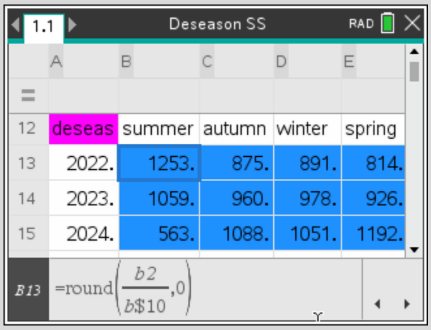
... continued

Constructing a spreadsheet to deseasonalise a time series (continued)

To calculate the deseasonalised quarterly sales figures:

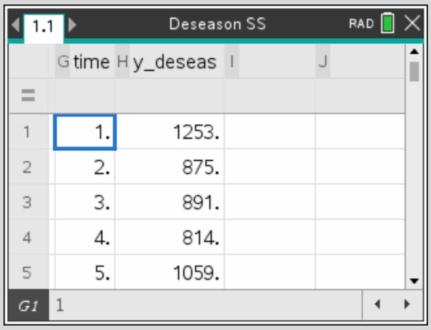
- In A12, enter the title ‘deseas’.
- In B12 to E12, enter the titles of the seasons.
- In B13, enter the formula **=round(b2/b$10,0)**
- Click in B13, then press [menu] > **Data** > **Fill**, and then press ▸ to select B13:B15.
- Press [enter] to fill the formula from B13 into B14:B15.
- Click in B13, then press [⇧shift] ▾ to select B13:B15.
- Press [menu] > **Data** > **Fill**, and press ▸ to select B13:E15.
- Press [enter] to fill formulae from B13:B15 into B13:E15.

The deseasonalised quarterly sales figures are now displayed, correct to the nearest whole number.



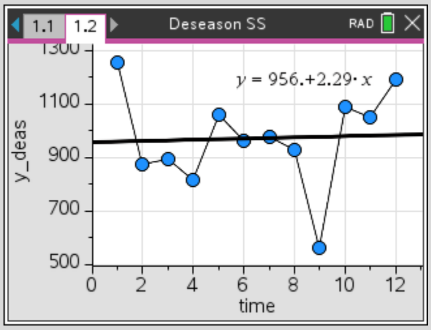
To prepare the deseasonalised data for plotting:

- In the heading for column G, enter the title ‘**time’.**
- In the heading for column H, enter the title ‘**y_deseas**’.
- In column G, enter the time periods 1 to 12.
- In column H, enter the deseasonalised quarterly sales data.



To construct a plot of the deseasonalised data, add a **Data & Statistics** page and then:

- Press [tab] to activate **Click to add variable** underneath the horizontal axis and select the variable ***time***.
- Press [tab] to activate **Click to add variable** to the left of the vertical axis and select the variable ***y_deseas***.
- Click in the window, then press [ctrl] [menu] and select the option **Connect Data Points**.

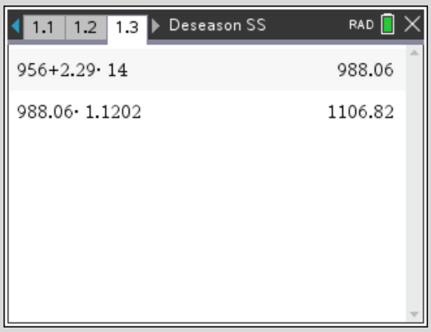


To overlay a least squares regression line:

- Press [menu] > **Analyse** > **Regression** > **Show Linear (a+bx)**

To estimate the quarterly sales for Autumn 2025 (i.e. for time period =14), add a **Calculator** page and then:

- Enter the expression **956+2.29×14**
- Enter the expression **ans × 1.1202** (SI (Autumn) = 1.1202)



Answer: The trend line has equation:

$\textit{deseasonalised sales} = 956 + 2.29 \times \textit{time period}$.

The trend line predicts in Autumn 2025 (time period = 14):

$\textit{deseasonalised sales} = 956 + 2.29 \times (14) \approx 988$ magazines.

$\textit{estimated actual sales} = 988.06 \times 1.1202 \approx 1107$ magazines.

Using a Notes page to deseasonalise a time series

The algorithm for deseasonalising a time series can be automated by constructing a **Notes** page, and by using some clever formulae. The process of constructing such a page is demonstrated below, using the data from the previous example (Yarra Bookstore) to illustrate the process.

The table below gives the number of magazines sold in each of the four seasons at the Yarra Bookstore for the years 2022 to 2024.

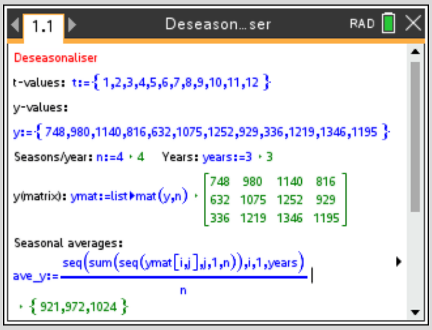
Year \ Season	*Summer*	*Autumn*	*Winter*	*Spring*
2022	748	980	1140	816
2023	632	1075	1252	929
2024	336	1219	1346	1195

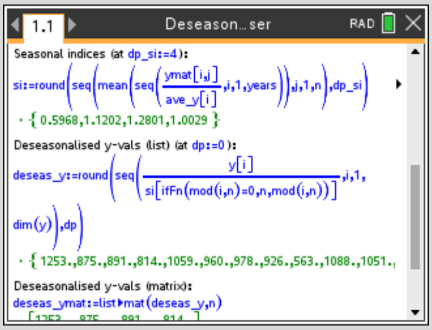
Set up a **Notes** page which can be used to deseasonalise a time series.

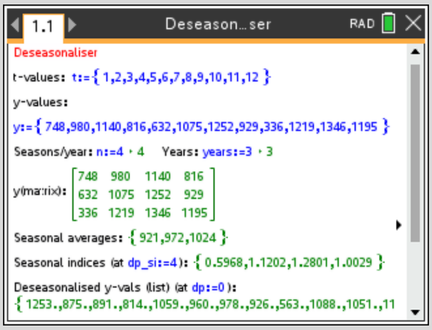
Add a **Notes** page and then:

- Enter the title text 'Deseasonaliser' as shown.
- Enter the text as shown.
- Click to the right of each line of text and press [ctrl] [M] to insert a Maths Box.
- Click inside each Maths Box, and enter the formulae as follows:
  - For 't-values:', enter ***t*:={1,2,3,4,5,6,7,8,9,10,11,12}**.
  - For 'y-values:', enter ***y*:={748,980,1140,816,632,1075, 1252,929,336,1219,1346,1195}**.
  - For 'Seasons/year:', enter ***n*:=4**.
  - For 'Years:' ***years*:=3**.
  - For '*y*(matrix):', enter ***ymat*:=list▸mat(*y*,*n*)**
  - For 'Seasonal averages:', enter $ave\_y := \dfrac{\text{seq}(\text{sum}(\text{seq}(ymat[i,j],j,1,n)),i,1,years)}{n}$.
  - For 'Seasonal indices', enter $si := \text{round}\left(\text{seq}\left(\text{mean}\left(\text{seq}\left(\dfrac{ymat[i,j]}{ave\_y[i]},i,1,years\right)\right),j,1,n\right),dp\_si\right)$.
  - For 'at ( ):', enter ***dp_si*:=4**
  - For 'Deseasonalised y-vals (list)', enter $deseas\_y := \text{round}\left(\text{seq}\left(\dfrac{y[i]}{si\left[\text{iffn}\left(\text{mod}(i,n)=0,n,\text{mod}(i,n)\right)\right]},1,\text{dim}(y),dp\right)\right)$
  - For 'at ( ):', enter ***dp*:=0**.
  - For 'Deseasonalised y-vals (matrix)', enter ***deseas_ymat*:=list▸mat(*deseas_y*,*n*)**.
  - Press [ctrl] [S] and save the file to the calculator.

*Note: The readability of this **Notes** page is greatly enhanced by hiding the input of some of the Maths Boxes, so the large formulas are not always visible. To do this, click in a Maths Box, and press [menu] **> Maths Box Options > Maths Box Attributes**, and select **Hide Input**. The final screen right shows the results of hiding the input of the longer formulae. Note also that the **list▸mat** command can be found via [catalog].*







3.5. Recursion & Financial Mathematics

3.5.1. Depreciation of assets

Comparing depreciation methods in the Calculator application

Consider a farm tractor with a purchase price of $60,000. The flat rate and reducing balance methods of asset depreciation after *n* years can be modelled by the recurrence relations $u_{n+1} = u_n - 6000,\ u_0 = 60000$ and $v_{n+1} = v_n \times 0.75,\ v_0 = 60000$, respectively.

Use a recursive process in the Calculator application to:

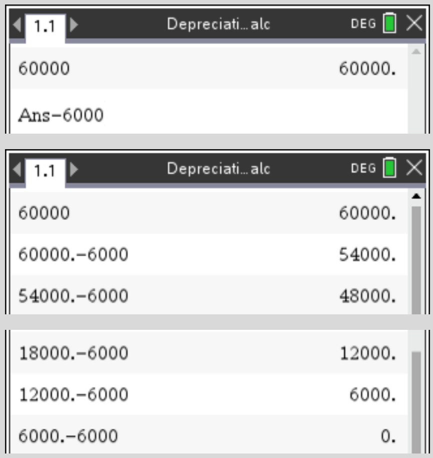
(a) Generate a sequence of the remaining value of the tractor after 0 to10 years using the flat rate method. Comment on how the value decreases over a period of 10 years.

(b) Generate a sequence of lists comparing the tractor's value over 0 to 10 years using flat rate and reducing balance methods and showing corresponding time periods. Compare the remaining value for each method after 3 years and after 10 years.

(a) To generate the values of the sequence with recursive rule $u_{n+1} = u_n - 6000,\ u_0 = 60000$, on a **Calculator** page:

- Enter **60000**
- Press [ctrl] [(–)] ([ans]) to input **Ans**, then [–] **6000** [enter]
- Press [enter] to generate each new value of the sequence.

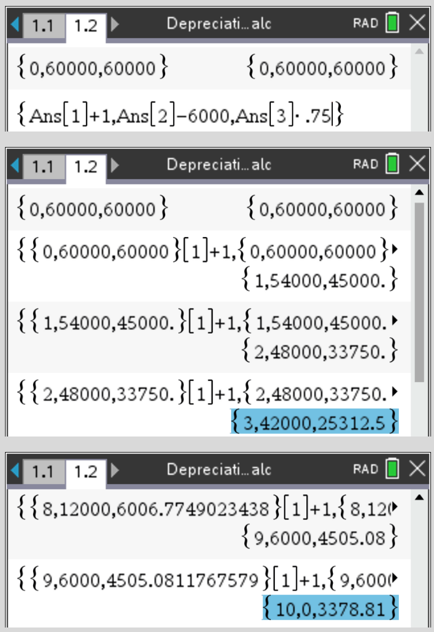
Answer: The flat rate method assumes that the value of the depreciating tractor decreases uniformly by $6,000 per year over an effective life of 10 years, to a value of $0.



(b) To generate values for time period *n*, $u_{n+1} = u_n - 6000$ and $v_{n+1} = v_n \times 0.75,\ u_0 = v_0 = 60000$, on a **Calculator** page:

- Enter **{0,60000,60000}**, being a list of three elements {time, flat rate value, reducing balance value}
- Enter **{Ans[1]+1,Ans[2]–6000,Ans[3]×0.75}**, pressing [ctrl] [(–)] ([ans]) to key in **Ans.** This input adds 1 to the first element of previous list, subtracts 6000 from the second element and multiplies the third element by 0.75.
- Press [enter] to generate each new term of the sequence.

Answer: After 3 years, the output {3,42000,25312.5} indicates that the flat rate and reducing balance remaining value are $42,000 and $25,312.50, respectively.
After 10 years, the remaining values are $0 and $3,378.81.



Comparing depreciation methods with Sequence graphing in the Graphs application

In the previous problem, the alternative depreciation methods are modelled by the recurrence relations $u_{n+1} = u_n - 6000$ and $v_{n+1} = v_n \times 0.75,\ u_0 = v_0 = 60000$.

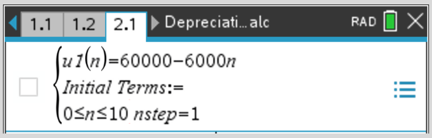
The corresponding explicit rules are $u_n = 60000 - 6000n$ and $v_{n+1} = 60000 \times (0.75)^n,\ n \geq 0$.

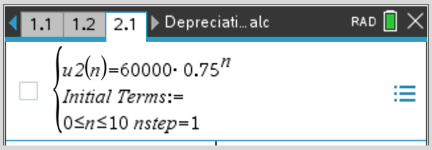
(a) Use sequence graphing to create a graphical display of the sequences.

(b) Hence use a tabular display to find the remaining value after 3, 6 and 9 years for each depreciation method.

(a) To set up **Sequence** graphs for the explicit rules $u_n = 60000 - 6000n$ and $v_{n+1} = 60000 \times (0.75)^n,\ n \in [0,10]$, on a **Graphs** page:

- Press [menu] > **Graph Entry/Edit > Sequence > Sequence**.
- Enter $\boldsymbol{u1(n) = 60000 - 6000n}$, **Initial Terms:=** (blank), $\boldsymbol{0 \leq n \leq 10}$ **nstep=1**, then press ▼.
- Enter $\boldsymbol{u2(n) = 60000 \times 0.75^n}$, **Initial Terms:=** (blank) $\boldsymbol{0 \leq n \leq 10}$ **nstep=1**, as shown.



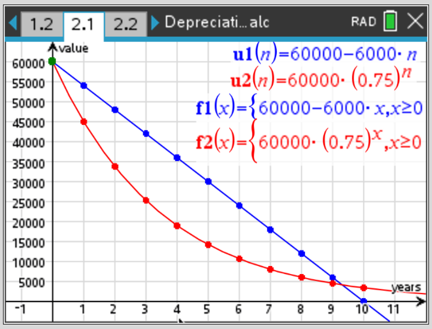


To adjust the window settings and add some enhancements:

- Press [menu] > **Window/Zoom > Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: **–2** XMax: **12** XScale: **1**
  YMin: **–5000** YMax: **65000** YScale: **5000**

To graph a continuous function that models the sequences:

- Press [menu] > **Graph Entry/Edit > Function**.
- Enter $\boldsymbol{f1(x) = 60000 - 6000x \mid x \geq 0}$ and $\boldsymbol{f2(x) = 60000 \times 0.75^x \mid x \geq 0}$.
- Press [ctrl] [menu] > **Hide/Show > Show Lined Grid**.
- Show multiple axes labels by hovering over an axis and pressing [ctrl] [menu] > **Attributes**. Select **Multiple Labels**.
- Rename the axes labels by clicking ( ) the label ***x*** and edit it to **year**. Similarly, edit the label ***y*** to **value**.

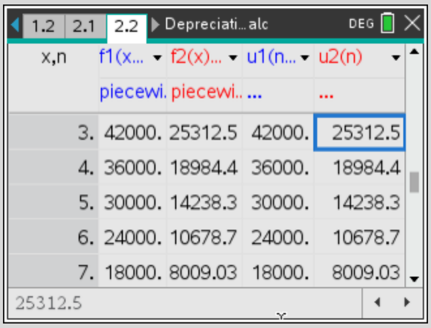


(b) To obtain a tabular display of the sequences u_n and v_n, from the **Graphs** page above:

- Press [ctrl] [T]

To view the table of values on a separate page:

- Press [doc▾] > **Page Layout > Ungroup**.
- Navigate to next page to view the table of values.



Answer: The table of values and the graphs confirm that
$u_3 = \$42,000,\ u_6 = \$24,000,\ u_9 = \$6,000$
$v_3 = \$25,312.50,\ v_6 = \$10,678.71,\ v_9 = \$4,505.08$

Note: *Sequence graphing accepts either an explicit rule, such as* $u_n = 60000 - 6000n$ *with notation* $u(n) = 60000 - 6000n$, *or the recurrence relation* $u_{n+1} = u_n - 6000,\ u_0 = 60000$ *with notation* $u(n) = u(n-1) - 6000$, Initial Terms := 60000.

Creating a Widget for unit cost depreciation using a dynamic Notes page

Suppose that a tractor dealership uses unit cost depreciation to determine the trade-in value based on the hours of use recorded on a tractor's hour meter. The trade-in value is modelled by the recurrence relation $w_{n+1} = w_n - R$ where $w_0 = \text{original value}$ and R is the unit cost rate.

The corresponding explicit rule is $w_{n+1} = w_0 - R \times n$ after n hours of use.

(a) Set up a dynamic Notes page to model unit cost depreciation.

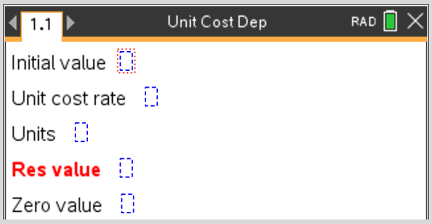
(b) Determine the trade-in value of a tractor with an hour meter reading of n = 2200 hours, if $w_0 = \$35{,}000$, R = \$8/hour. Determine hours of use for the tractor to have zero value.

(c) Assume that a tractor that gets 450 hours of use per year. Determine its trade-in value after 3, 5 and 10 years if $w_0 = \$60{,}000$, R = \$12/hour. Edit the fifth Maths Box formula to determine hours of use for the value of the tractor to be less than \$25,000.

(d) Save the document as a Widget to carry out for similar calculations in the future.

(a) To set up unit cost depreciation, on a **Notes** page:

- Enter the text labels, as shown.
- Place the cursor next to the label 'Initial value'. Press ctrl M to insert a **Maths Box**.
- Similarly, insert a **Maths Box** next to every other label.

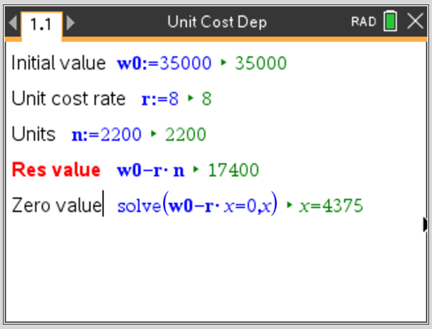


(b) To determine the remaining (trade-in) value for n = 2200 hours, if $w_0 = \$35{,}000$, R = \$8/hour:

- In the text top **Maths Box**, key in ***w*0:=35000**, pressing ctrl [templates] (:=) for the assign symbol, then press enter.
- Similarly, enter as shown in the next 3 Maths Boxes.

To find hours of use for the tractor to have zero value:

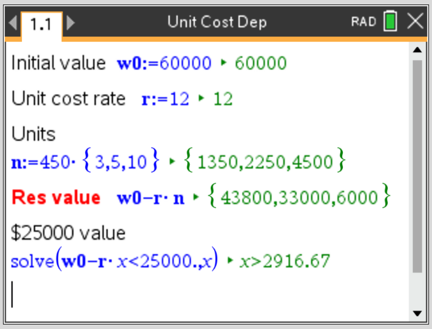
- In the bottom Maths Box, press menu > **Calculations** > **Algebra** > **Solve**. Enter **solve(*w*0 – *r* × *x* = 0, *x*)**, as shown.



Answer. The trade-in value is \$17,400 for 2200 hours of use. The nominal value will be zero after 4375 hours of use.

(c) To determine the remaining value for n = 450 hours/year, if $w_0 = \$60{,}000$, R = \$12/hour after 3, 5 and 10 years:

- Edit the inputs in the **Maths Boxes** 1, 2, 3 and 5 to **w0:=60000**, ***r*:=8**, ***n*:=450×{3,5,10}**, and **solve(*w*0 – *r*×*x* < 25000.0, *x*)** as shown.



Answer. The trade-in values after 3, 5 and 10 years (1350, 2250 and 4500 hours) are \$43,800, \$33,000 and \$6,000
The value will be less than \$25,000 after 2,917 hours of use.

... continued

Creating a Widget for unit cost depreciation using a dynamic Notes page (continued)

(d) To save this document as a **Widget** to carry out similar calculations in the future:

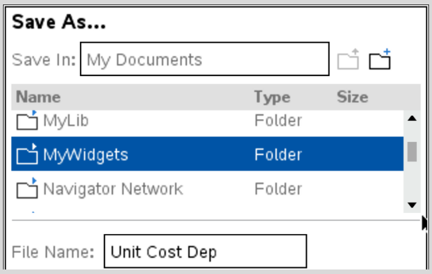
- Press doc▾ > **File** > **Save As …** and navigate to the **MyWidgets** folder. Enter the file name and click **Save**.

To open the **Widget** within an opened document:

- Press ctrl doc▾ (+page) > **Add Widget**. Select the file.

To open the **Widget** as a new document:

- Press ⌂on > **New** > **Add Widget**. Select the file.



Modelling depreciation methods in the Lists & Spreadsheet application

Note: *This section illustrates a variety of ways that sequences can be generated in the Lists & Spreadsheet application using both recurrence relations and explicit rules.*

An agricultural contractor purchases a new hay baler for $84,000. The following instructions are used to model three depreciation methods for this asset on a Lists & Spreadsheet page.

(a) Generate a sequence of integers from 0 to 15 years by filling down a recursive formula.

(b) Use the '*seq*' command to generate a sequence that models flat rate depreciation with recursive and explicit rules $u_0 = 84000,\ u_{n+1} = u_n - 7000$ and $u_n = 84000 - 7000n$.

(c) Use the 'Generate Sequence' command to generate a sequence that models reducing balance depreciation with rules $v_0 = 84000,\ v_{n+1} = 0.8v_n$ or $v_n = 84000\times(0.8)^n$.

(d) If a unit cost depreciation rate, *R*, of $2.20 per round hay bale produced is applied, the remaining asset value after *n* years can be modelled by the recurrence relation $w_0 = 84000,\ w_{n+1} = w_n - R\times b$, where *R* = $2.20 and *b* is the number of bales produced that year. Assume that the same number of bales are produced every year.
Generate sequences for this rule with *b* = 3000 and *b* = 5000, using a recursive method.

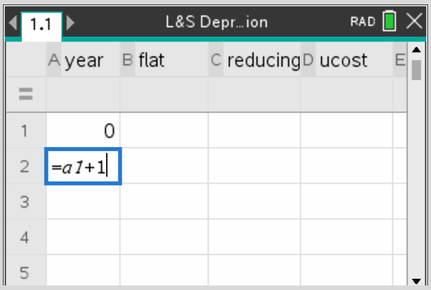
To set up the spreadsheet, on a **Lists & Spreadsheet** page:

- In the column A, B, C and D heading cells respectively, enter the headings **year**, **flat**, **reducing** and **ucost**.

(a) To recursively generate the sequence of years, 0 to 15:

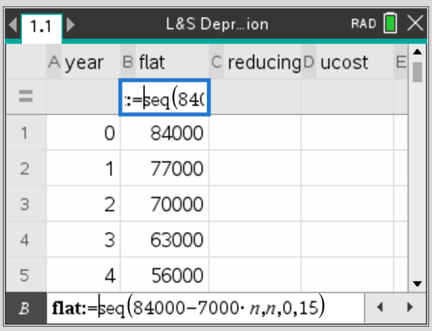
- Enter **0** in cell A1, then enter the formula **=*a*1+1** in cell ***A*2**.
- Navigate to cell A2 and press ctrl menu > **Fill**.
- Press ▾ down to cell A16, then press enter.

Note*: The cell reference* ***a1*** *is relative to the cell location. When filled down, it renews to '****=a2+1****' , '****=a3+1****' etc.*



(b) To generate the sequence with the explicit rule $u_n = 84000 - 7000n, n = \{0,1,...,15\}$ using the **seq** command:

- Navigate to the column B formula cell.
- Press 📖 1 S and navigate to **seq**. The syntax is **seq**(Expr, Var, Low, High [,Step]). The default Step = 1.
- Enter the formula $\mathbf{flat := seq(84000 - 7000n, n, 0, 15)}$.

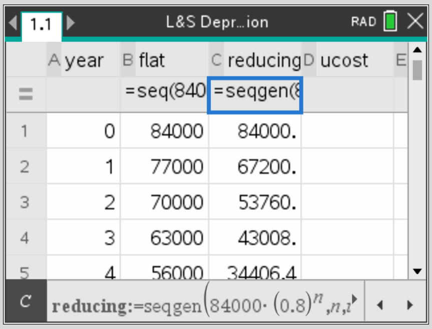


Modelling depreciation methods in the Lists & Spreadsheet application (continued)

(c) To generate the sequence with explicit rule $v_n = 84000 \times (0.8)^n$ using **Generate Sequence**:

- Navigate to the column C formula cell.
- Press [menu] > **Data** > **Generate sequence**.

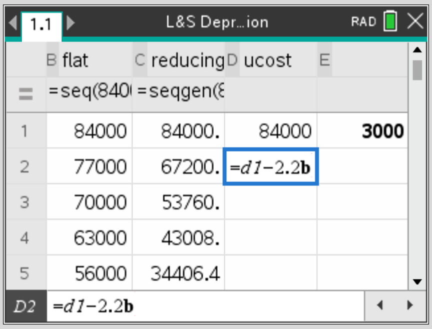
In the dialog box that follows, enter the values:
Formula: $u(n) =$ $\mathbf{84000 \times 0.8^n}$, Initial Terms: (not required)
n0: **0**, nMax : **15**, nStep = **1**, Ceiling Value: (leave blank)

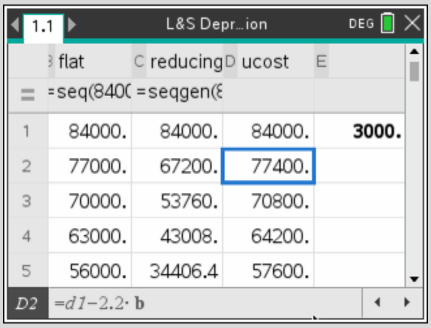


***Note: Generate Sequence** also accepts the recursive rule, $v_{n+1} = 0.8v_n, v_0 = 84000$ by changing the first two fields to Formula: u(n) =* $\mathbf{0.8 \times u(n-1)}$*, Initial Terms:* **84000**

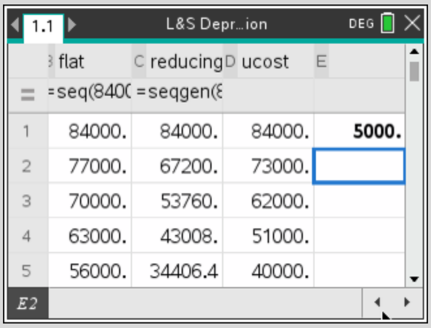
(d) To recursively generate the sequence with rule $w_0 = 84000,\ w_{n+1} = w_n - R \times b$, R = $2.20 and b can vary:

- Navigate to cell E1 and enter ***b*:=3000**.
  This represents the number of bales/year.
- Enter **84000** in cell D1 and **=*d*1– 2.2*b*** in cell D2.
- Navigate to cell D2 and press [ctrl] [menu] > **Fill**.
- Press ▼ down to cell D16, then press [enter].





- Change the value of ***b*** by editing cell E1 to ***b*:=5000**.



Graphing and comparing depreciation methods in the Data & Statistics application

Use the sequences generated in the previous problem to model graphically flat rate, reducing balance and unit cost depreciation on a hay baler purchased for $ 84,000.

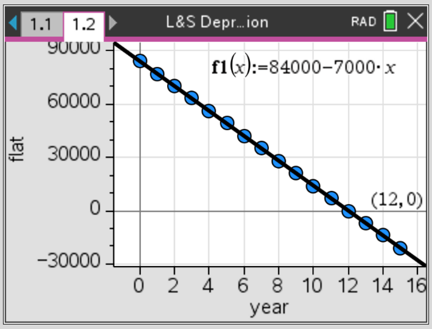
To graph the *flat rate* depreciation from the previous problem:

- Press ctrl doc▾ > **Add Data & Statistics**.
- Press tab and select **year** on the horizontal axis.
- Press tab and select **flat** on the vertical axis.

To add a continuous function that models the sequence:

- Press menu > **Analyse** > **Plot Function**.
- In the text box that follows, enter $f\,1(x)$**:=84000 – 7000**x.

Note: *The remaining value is zero after 12 years.*



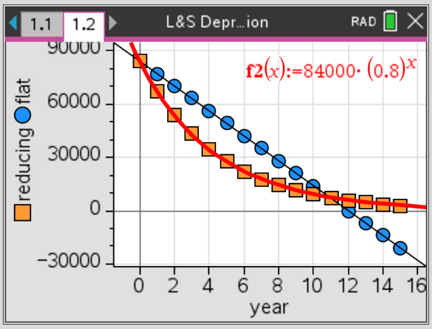
To graph the *reducing balance* depreciation from the previous problem, on the **Data & Statistics** page:

- Press menu > **Plot Properties** > **Add Y Variable**.
- Select **reducing** from the variable list that follows.

To add a continuous function that models the sequence:

- Press menu > **Analyse** > **Plot Function**.
- In the text box that follows, enter $f\,2(x)$**:=84000×(0.8)**^{x}.

Note: *The remaining values are almost equal after 11 years.*



To graph the unit cost depreciation:

- Press menu > **Plot Properties** > **Add Y Variable**.
- Select **ucost** from the variable list that follows.

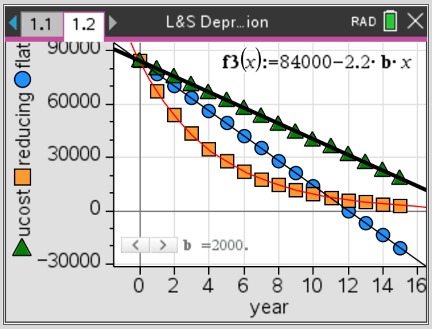
To add a continuous function that models the sequence:

- Press menu > **Analyse** > **Plot Function**.
- In the text box, enter $f\,3(x)$**:=84000 – 2.2**b**×**x.

To add a slider controlling the value of b:

- Press menu > **Actions** > **Add Slider**.
  In the dialog box that follows, enter the values:
  Variable: ***b***, Value: **1000**, Minimum: **500**,
  Maximum: **5000**, Step Size: **500**, Display digits: **Float 4**
- Use the slider to dynamically change the value of ***b*** and observe how the value of b affects the number of years it takes for the hay baler to have a value of $0 by the unit cost method.

Notes:
1*. The slider controls* ***b*** *value in the column D formulas.*
2. *The spreadsheet data could alternatively be graphed as* ***Scatterplots*** *on a* ***Graphs*** *page.*



3.5.2. Compound interest investments and loans

Modelling and analysing a compound interest investment using a recurrence relation

(a) Use the **Notes** and **Lists & Spreadsheet** applications to set up an editable and interactive document to numerically model and analyse a compound interest investment or loan.

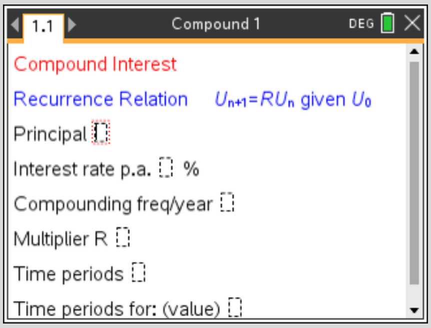
(b) Hence determine the final balance and interest earned on a principal sum of $40,000 deposited for 6 years at 5% pa if interest is compounded **(i)** annually, **(ii)** monthly.

(c) For part **(b)** above with interest compounded monthly, determine the minimum number of time periods for the closing balance to exceed $50,000.

(a) To set up input fields for the investment, on a **Notes** page:

- Enter the text shown, with input captions, **Principal** etc.
- Place the cursor to the right of the caption, **Principal** and press [ctrl] [M] to insert a **Maths Box**.
- Similarly, insert a **Maths Box** next to each of the other input captions.

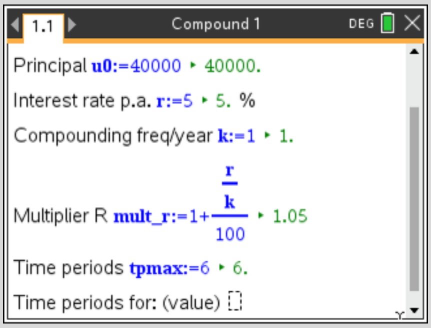
Note: *Press [template key] to select the subscript template. Press [?!▸] to select the % symbol.*



(b) (i) To enter the investment details in the input fields:

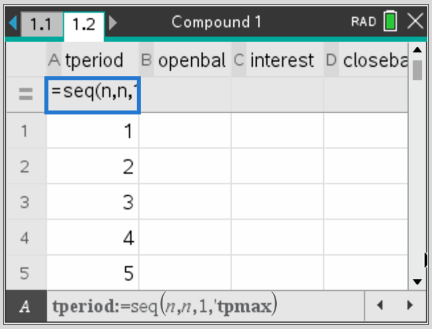
- In the **Maths Box** next to **Principal**, key in ***u*0:=40000** by pressing [ctrl] [template key] (:=) to access the **assign** command, then press [enter] to activate the **Maths Box**.
- Similarly, in each of the other **Maths Boxes**, enter the inputs as shown. Leave the last **Maths Box** blank for now.

Note: *To key in 'underscore' symbol in **mult_r**, press* [ctrl] [␣].



To set up a table showing time period, opening balance, interest, closing balance, on a **Lists & Spreadsheet** page:

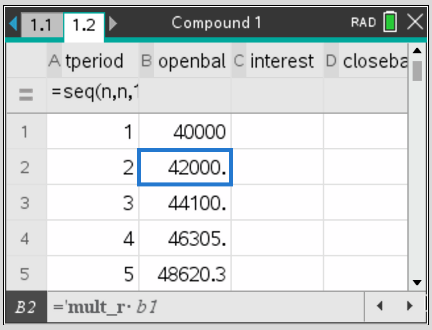
- In the heading cells, enter the column headings as shown.
- Navigate to the column A formula cell.
- Press [book] [S] and navigate to **seq**. The syntax is **seq**(Expr, Var, Low, High [,Step]). The default Step = 1.
- Enter the formula $=\mathbf{seq}(n,n,1,\mathbf{'tpmax})$, by pressing [var] to select the variable **tpmax**.



To calculate the opening balance at the start of each period by recursively filling down a formula:

- In cell B1, enter the formula, $='u0$, pressing [var] to select the variable ***u*0**.
- In cell B2, enter the formula, $=\mathbf{'mult\_r}\times b1$, pressing [var] to select the variable **mult_r**.
- Navigate to cell B2, press [ctrl] [menu] > **Fill**. Press ▼ to cell B6 then [enter]. This fills the formula down to cell B6.

Note: *The cell reference **b1** is relative to the cell location. When filled down, it renews to* $=\mathbf{'mult\_r}\times b2$ *etc.*



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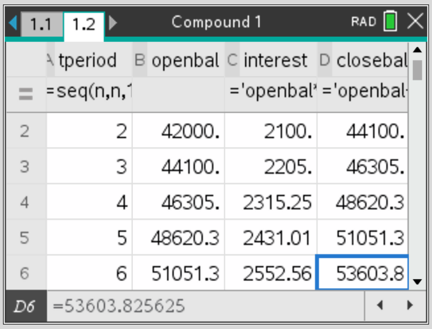
Modelling and analysing a compound interest investment using a recurrence relation (cont.)



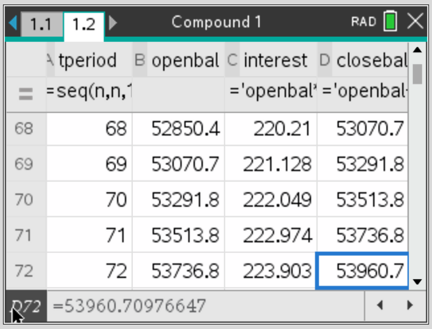
To calculate the interest for each time period:

- In the column C formula cell enter the formula, $=\dfrac{\text{'openbal}\times\text{'r}}{\text{'k}\times 100.}$, pressing [var] to select the variables **openbal**, *r* and *k*.

To calculate the closing balance for each time period:

- In the column D formula cell enter the formula, $=\text{'openbal}+\text{'interest}$, pressing [var] to select the variables **openbal** and **interest**.

Answer. After 6 years, the closing balance is $53,603.83.



(b) (ii) Approach 1

To recalculate the formulas for interest compounded monthly, on the **Notes** page 1.1:

- Edit the compounding frequency **Maths Box** to ***k*:=12**.
- Edit the 'Time periods' **Maths Box** to **tpmax:=6×12**.
- On the **Lists & Spreadsheet** page 1.2, navigate to cell B2 and fill the formula down to cell B72.

Answer. After 6 years with interest compounded monthly, the closing balance is $53,960.71 ($356.88 more than **(b)(i)**.)

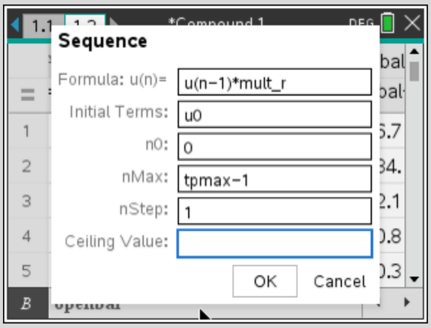
Note: *Shortcuts to navigate the* ***Lists & Spreadsheet*** *application.*
[ctrl][1]: *Move to end of list/page down* [ctrl][7]: *Move to start of list/page up*
[ctrl][3]: *Page down* [ctrl][9]: *Page up* [ctrl][G]: *Go to cell (enter the cell reference)*

(b) (ii) Alternative approach

- Edit the compounding frequency **Maths Box** to ***k*:=12**.
- Edit the 'Time periods' **Maths Box** to **tpmax:=6×12**.

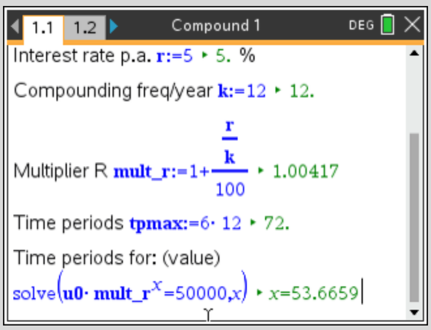
To generate the sequence $u_{n+1} = R\times u_n, u_0 = 40000$, on the **Lists & Spreadsheet** page 1.2:

- Navigate to the column B formula cell.
- Press [menu] > **Data** > **Generate Sequence**.
- In the dialog box that follows, enter the values as shown.
- If a **Data Loss** warning appears, click **OK** to proceed.



Note: *This approach automates calculations when there is a change to the time period or compounding frequency. However, note the alternative notation for the Formula field.*

(c) To determine the minimum number of time periods for the closing balance to exceed $50,000, on the **Notes** page 1.1:



- In the last **Maths Box**, next to 'Time Periods for:', press [menu] > **Calculations** > **Algebra** > **Solve**.
- Enter $\textbf{solve}(\boldsymbol{u0}\times\textbf{mult\_r}^x = 50000, x)$, pressing [var] to select the variables ***u*0** and **mult_r**.

Answer. The balance will exceed $50,000 after 54 months. This is confirmed by analysing the table on page 1.2.

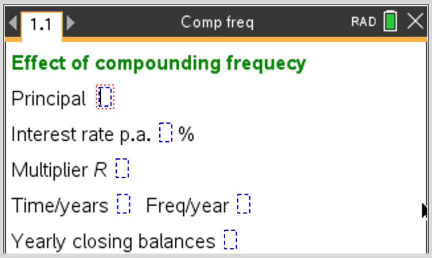
Comparing the effect of different compounding frequencies on an investment or loan

(a) Use the **Notes** application to set up an interactive document to model and analyse the effect of different compounding frequencies on an investment.

(b) Determine the effect of earning 20% annual interest on an initial investment of $1,000 after 10 years, if interest is compounded annually, quarterly, monthly and daily.

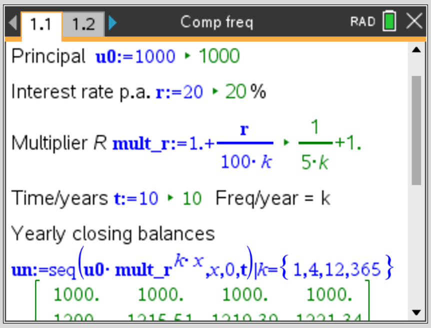
(a) To set up input fields for the investment, on a **Notes** page:

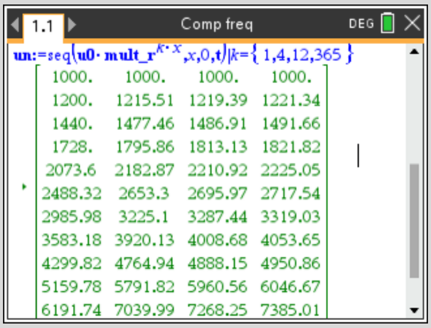
- Enter the text shown, with input captions, **Principal** etc.
- Place the cursor to the right of the caption, **Principal** and press [ctrl] [M] to insert a **Maths Box**.
- Similarly, insert a **Maths Box** next to each of the other input captions, pressing [?!▸] to select the **%** symbol.



(b) To enter the investment details and obtain a tabular display for the different compounding frequencies:

- In the **Maths Box** next to '**Principal**', key in ***u*0:=1000** by pressing [ctrl] [templates] (:=) to access the **assign** command, then press [enter] to activate the **Maths Box**.
- Similarly, enter as shown in each of the other **Maths Boxes**, pressing [ctrl] [␣] for *'underscore' in* ***mult_r***.





Answer: For an initial investment of $1000 under the above conditions:

- If interest is compounded *annually*, the balance after 10 years is $6191.74.
- If interest is compounded *quarterly*, the balance after 10 years is $7039.99.
- If interest is compounded monthly, the balance after 10 years is $7268.25.
- If interest is compounded *daily*, the balance after 10 years is $7385.01.

Analysing compounding frequencies using an explicit rule with the Graphs application

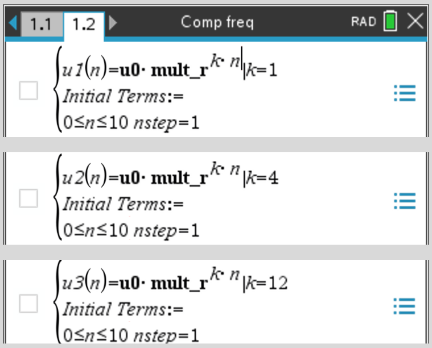
Model the previous problem using a graphical display to show the impact of 20% annual interest on a $1,000 investment over 10 years, compounded annually, quarterly, monthly and daily. Compare the interest earned after 10 years for the different compounding frequencies.

To graphically model different compounding frequencies, using ***u*0** and **mult_r**, assigned in the previous problem:

- Open the document from the previous problem.
- Press [ctrl] [doc▾] (+page) > **Add Graphs**.
- Press [menu] > **Graph Entry/Edit** > **Sequence** > **Sequence**.

To model annual compounding frequency as sequence ***u*1(*n*)**:

- Key in the sequence graph details as shown, then press ▼.
- If prompted to insert a ***k*** slider, click **Cancel** to dismiss.



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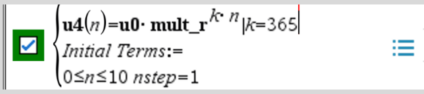
Analysing compounding frequencies using an explicit rule with the Graphs application (cont.)

To model compounding frequency of 4, 12 and 365/year:

Similarly, key in the details for sequences $\boldsymbol{u2(n)}$, $\boldsymbol{u3(n)}$ and $\boldsymbol{u4(n)}$, as shown, pressing enter at the conclusion.



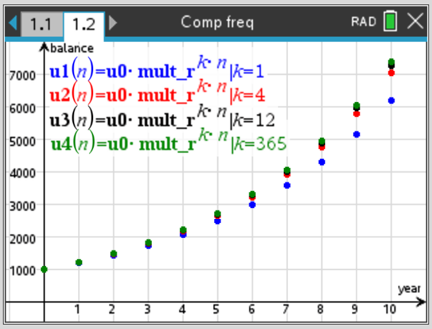
To set an appropriate graphing window:

- Press menu > **Window/Zoom > Window Settings.**

In the dialog box that follows, enter the values:

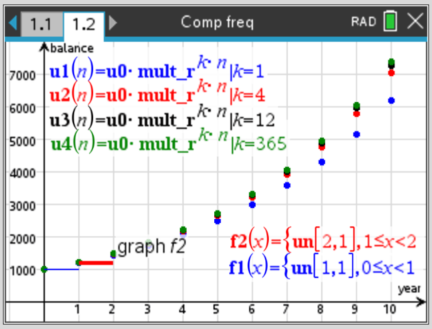
XMin: **–1.1** XMax: **11** XScale: **1**
YMin: **–600** YMax: **8000** YScale: **1000**

- Press ctrl menu > **Hide/Show > Show Lined Grid**.
- Show multiple axes labels by hovering over an axis and pressing ctrl menu > **Attributes**. Select **Multiple Labels**.
- Edit axis label x to **year** and edit axis label y to **balance**.



To show annual compounding as a step function:

- Press menu > **Graph Entry/Edit > Function**.
- Key in $f1(x) = un[1,1] \mid 0 \le x < 1$
- Select the text '$un[1,1] \mid 0 \le x < 1$' by placing the cursor at the end and pressing ◀ while holding down the ⇧shift key.
- Copy the text by pressing ctrl C. Press ▼ to $f2(x)$.
- Press ctrl V, then edit text to $f2(x) = un[2,1] \mid 1 \le x < 2$.

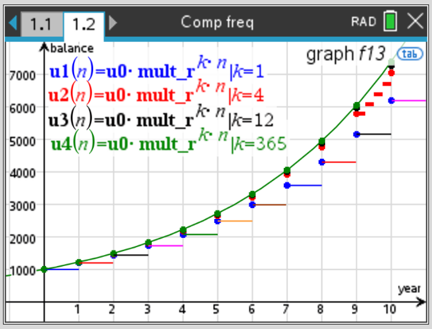


To complete the step function:

- Press ▼ to $f3(x)$, then press ctrl V to paste the text.
- Edit text to $f3(x) = un[3,1] \mid 2 \le x < 3$
- Repeat as above until $f11(x) = un[11,1] \mid 10 \le x < 11$

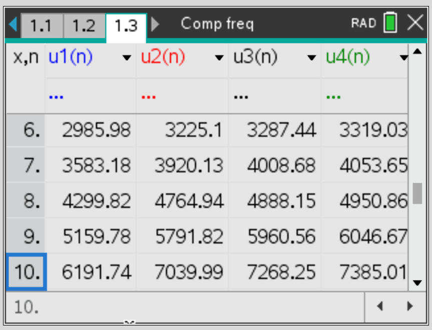
To approximate daily compounding with a continuous curve:

- Enter $f11(x) = u0 \times mult\_r^{k \times x} \mid k = 365$
- Press menu > **Actions > Hide/Show** to hide labels.

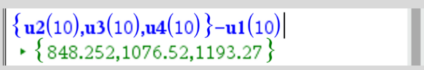


To show a table of values for the **Sequence** graphs:

- Press ctrl T to toggle showing/hiding a table of values.
- Press doc▾ > **Page Layout > Ungroup**.
- On the table page 1.3, click the drop-down menu icon in the heading cells and select $\boldsymbol{u1}$, $\boldsymbol{u2}$, $\boldsymbol{u3}$, $\boldsymbol{u4}$, as shown.



x,n	u1(n)	u2(n)	u3(n)	u4(n)
	...	...	...	...
6.	2985.98	3225.1	3287.44	3319.03
7.	3583.18	3920.13	4008.68	4053.65
8.	4299.82	4764.94	4888.15	4950.86
9.	5159.78	5791.82	5960.56	6046.67
10.	6191.74	7039.99	7268.25	7385.01



Answer. After 10 years, the closing balances for annual, quarterly, monthly and daily compounding are \$6,191.74, \$7,039.99, \$7,268.25 and \$7,385.01, respectively. Compared to annual compounding, quarterly, monthly and daily yields an additional \$848.25, \$1,076.52 and \$1,193.27, respectively.

Texas Instruments

Nominal and effective interest rates for different compounding frequencies

In the previous problem it was shown that when a nominal interest rate of 20% p.a. is applied to a $1000 investment, the balances after 1 year for annual, quarterly, monthly and daily compounding frequencies are $1,200, $1,215.51, $1,219.39 and $1,221.34, respectively.

(i) Determine the effective interest rates for each compounding frequency, to three decimal places, using:

(i) first-principles definition,

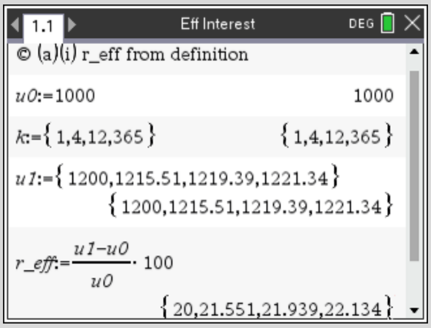
(ii) the formula $r_{eff} = \left(1+\frac{r_n}{k}\right)^k - 1$, where r_n is the nominal interest rate expressed as a decimal and k is the compounding frequency.

(iii) effective interest rate command in the Finance menu.

(ii) Show that if the effective interest rates are applied to annual compounding, the balances after 10 years will be $6,191.74, $7,039.99, $7,268, and $7,385.01. This result is equivalent to applying the nominal rate of 20% to annual, quarterly, monthly, and daily compounding frequencies.

(a)(i) To find the effective rates from the first-principles definition, on a **Calculator** page:

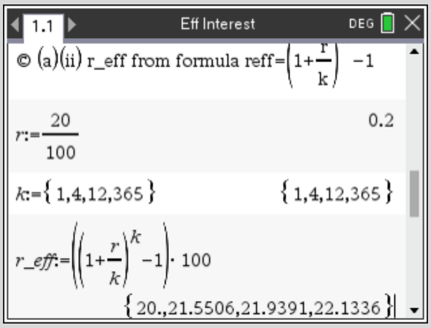
- Enter ***u*0:=1000** by pressing ctrl [templates] (:=) to key in the **assign** command.
- Enter as shown, the compounding frequencies, ***k***, the balances after 1 years, ***u1,*** and the first-principles formula $\boldsymbol{r\_eff} := \frac{\boldsymbol{u1-u0}}{\boldsymbol{u0}}\times\mathbf{100}$, pressing ctrl [space] for underscore.



Answer. r_{eff}={20,21.551,21.939,22.134} for k = {1,4,12,365}

To determine the effective interest rates using the formula, on a **Calculator** page:

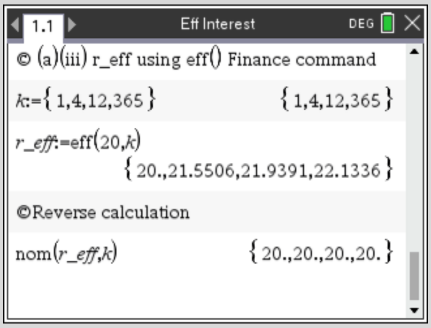
- Enter as shown, the nominal rate expressed as a decimal, ***r***, the compounding frequencies, ***k***, and the formula $\boldsymbol{r\_eff} := \left(\mathbf{1}+\frac{\boldsymbol{r}}{\boldsymbol{k}}-\mathbf{1}\right)^{\boldsymbol{k}}\times\mathbf{100}$.



Answer. r_{eff}={20,21.551,21.939,22.134} for k = {1,4,12,365}

To determine the effective interest rates using the **eff() Finance** command, on a **Calculator** page:

- Enter the compounding frequencies, ***k***, as shown.
- Enter ***r_eff*:=eff(20,*k*)**. To paste in the **eff()** command, press menu > **Finance** > **Interest Conversion** > **Effective Interest Rate**. The syntax is **eff(**nominal rate, compounding frequency per year).



Answer:. r_{eff}={20,21.551,21.939,22.134} for k = {1,4,12,365}

To test the reverse calculation using the **nom()** command:

... continued

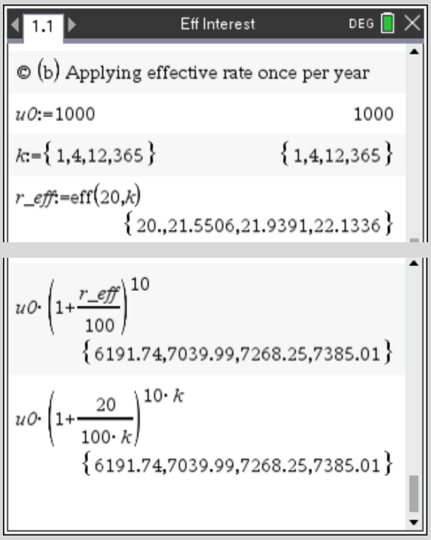
- Press [menu] > **Finance** > **Interest Conversion** > **Nominal Interest Rate**, then enter **nom(*r_eff*,*k*)**. The syntax is **nom(**effective rate, compounding frequency per year**)**.

Nominal and effective interest rates for different compounding frequencies (continued)

(b) To show that compounding the effective rates annually is equivalent to compounding the nominal rates ***k*** times per year:

- Enter as shown, the initial value, ***u*0**, the compounding periods, ***k***, and the effective rates, ***r_eff***.
- Enter $u0 \times \left(1 + \frac{r\_eff}{100}\right)^{10}$ to calculate the balance after 10 years of compounding the effective rates annually.
- Enter $u0 \times \left(1 + \frac{20.}{100k}\right)^{10k}$ to calculate the balance after 10 years of compounding a 20% nominal rate, ***k*** times/year.

Answer: The results of the two calculations are identical, showing the required equivalence.



Note: *The above calculations could also be performed in* ***Maths Boxes*** *in the* ***Notes*** *application.*

3.5.3. Reducing balance loans

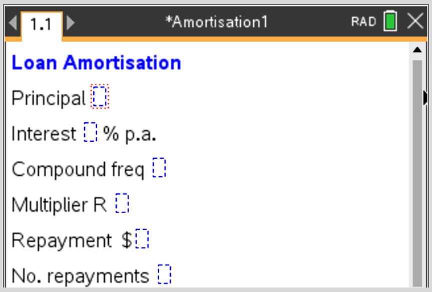
Analysing the amortisation of a loan on a step-by-step basis

Laila borrows $30,000 to buy a car. The interest rate is 9% p.a., compounded monthly.
She is required to make minimum monthly repayments of $954 per month for 3 years.
Use a table to investigate the monthly amortisation schedule, showing the payment made, the amount of interest paid, the reduction in the principal and the balance of the loan.

(a) To set up an editable amortisation schedule analyser, on a **Notes** page:

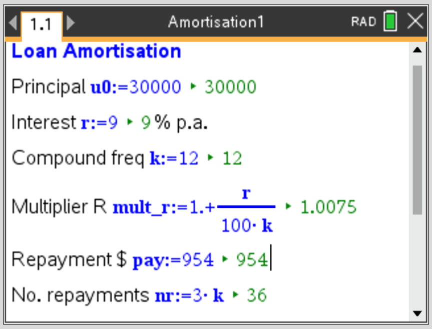
- Enter the text shown, with input captions, **Principal** etc.
- Place the cursor to the right of the caption, **Principal** and press [ctrl] [M] to insert a **Maths Box**.

Similarly, insert a **Maths Box** next to each of the other input captions, pressing [?!▸] to select the **%** symbol.



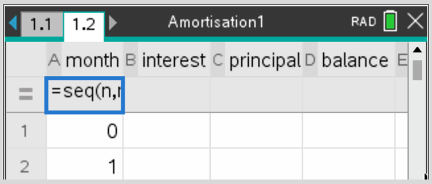
To enter the inputs for the amortisation schedule analyser:

- In the **Maths Box** next to '**Principal**', enter ***u*0:=30000**, pressing [ctrl] [template] (:=) to key in the **assign** command.
- Enter the assigned values in the remaining **Maths Boxes** as shown, pressing [ctrl] [␣] for 'underscore' in **mult_r**.



To set up the amortisation table:

- Press [ctrl] [doc▾] (+page) > **Add Lists & Spreadsheet**.
- In the columns A, B, C and D heading cells, enter the headings as shown.
- In the column A formula cell, enter **=seq(*n*,*n*,0,*nr*)**.



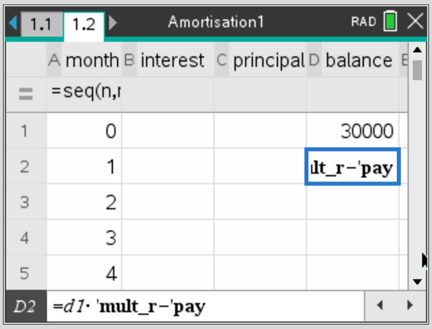
Approach 1 to generating the loan balance sequence

Note: *This approach to generating the sequence with recurrence relation* $u_{n+1} = R \times u_n - d$, *where R=**mult_r**, d=**pay**, involves filling down a recursive formula.*

To model the monthly balances with a recurrence relation:

- In cell D1, enter the formula, **='*u*0**, then in cell D2, enter **=*d*1×'mult_r – 'pay**, by pressing [var] to select the stored variables ***u*0**, **mult_r** and **pay**.
- Navigate to cell D2, then press [ctrl] [menu] > **Fill**.
- Press ▼ to fill down to cell D37, then press [enter].

Note 1. The cell reference d1 is relative to the cell location. When filled down, it renews to ***=d2×'mult_r – 'pay*** *etc.*
Note 2. An apostrophe in front (e.g. 'u0) indicates a variable reference. This appears when a variable is selected from [var].



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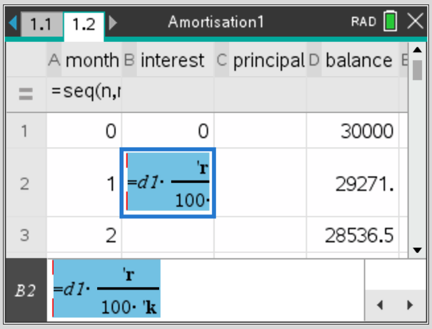
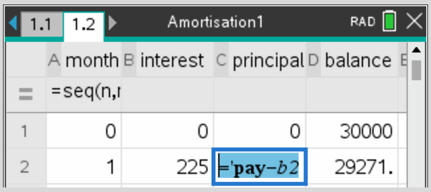
Approach 1 (continued)



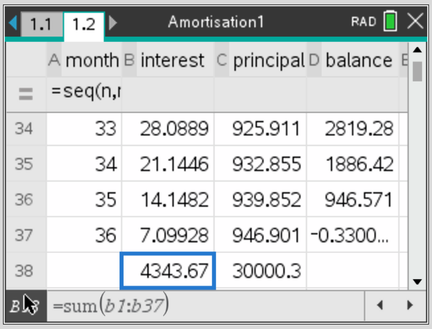
To model the monthly interest payment:

- Enter **0** in cell B1.
- In cell B2, enter the formula, $=d1\times\frac{'r}{100\times'k}$, by pressing [var] to select the variables ***r*** and ***k***.
- Navigate to cell B2, then press [ctrl] [menu] > **Fill**.
- Press ▼ to fill down to cell B37, then press [enter].



To model the monthly payment towards the principal:

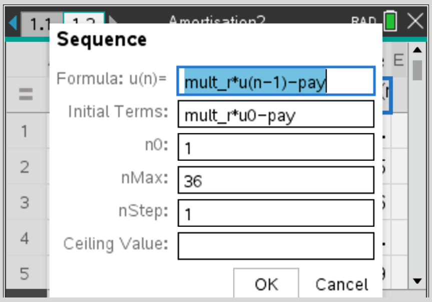
- Enter **0** in cell C1.
- In cell C2, enter the formula, = **'pay** – ***b*2**, by pressing [var] to select variable, **pay**.
- Navigate to cell B2, then press [ctrl] [menu] > **Fill**.
- Press ▼ to fill down to cell B37, then press [enter].
- Navigate to cell B38, press [menu] > **Data** > **List Maths** > **Sum of Elements** and enter the formula, **=sum(*b*1:*b*37)**, by pressing [?!▸] to select the colon, **:**.
- Similarly, in cell C38 enter the formula, **=sum(*c*1:*c*37)**.



Answer. The balance is $0 after 36 months (rounded to the nearest dollar) and the sum of repayments of principal is $30,000, as expected. The total interest paid is $4,344.

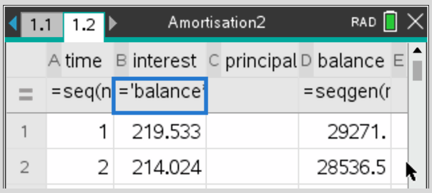
Approach 2 to generating the loan balance sequence

Note: *For this approach, the* ***Generate Sequence*** *command is used for the loan balance sequence. For the recurrence relation of the form* $u_{n+1}=R\times u_n-d$*, the notation is* $\boldsymbol{u(n)=R\times u(n-1)-d}$.



To model the monthly balances with a recurrence relation of the form $u_{n+1}=R\times u_n-d$:

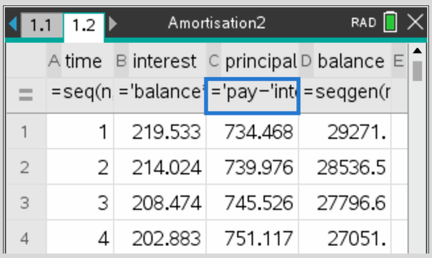
- Enter the headings as shown above.
- In the column A formula cell, enter **=seq(*n*,*n*,1,*nr*)**
- Navigate to the column D formula cell (second from the top) and press [menu] > **Data** > **Generate Sequence**.
- In the dialog box that follows, enter the inputs as shown.



To model the monthly interest payment:

- In the column B formula cell, enter the formula $='\mathbf{balance}\times\frac{'r}{100\times'k}$, pressing [var] to select variables, **balance**, ***r*** and ***k***.

To model the monthly payment towards the principal:



- In the column C formula cell, enter the formula, **='pay – 'interest**, pressing [var] to select the variables, **pay** and **interest**.

Answer. The table is the same as for Approach 1, starting from the end of the first month. total interest paid is $4,344.

Analysing a loan and amortisation table using 'amortTbl()' and other Finance commands

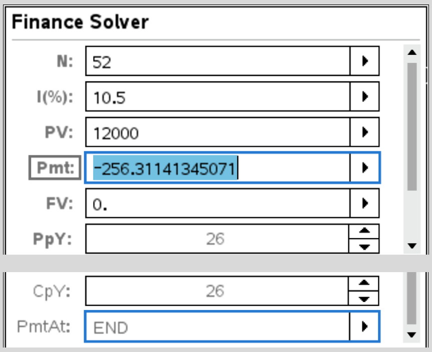
Consider a $12,000 loan with an interest rate of 10.5% per annum, compounded fortnightly.

(a) If the loan is to be paid off in 2 years, determine the fortnightly loan repayments.
(b) Create an amortisation table for the loan.
(c) Determine the interest paid in the first 12 months and total interest paid over the 2 years.

(a) To determine the fortnightly loan repayment amount, add a **Notes** page to a new document, then:

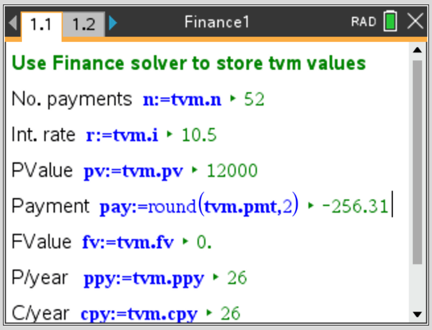
- Press [menu] > **Calculations** > **Finance** > **Finance Solver**.
- On the dialog box, key in the values shown, except for the **Pmt** field. Press [tab] to navigate between fields.
- When all other fields have values allocated, navigate to the **Pmt** field and press [enter] to calculate the payment.
- Press [var] to confirm that the values for each field have been stored as ***tvm*** variables, ***tvm.n***, ***tvm.i***, ***tvm.pv*** etc.

Answer. The fortnightly repayments are $256.31.



(b) To display the values of the fields, on the **Notes** page:

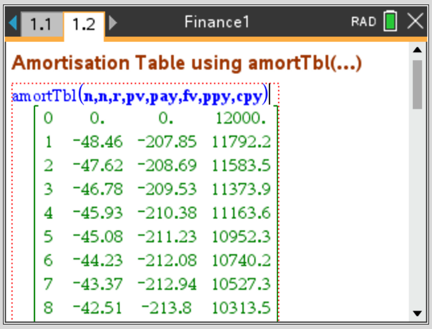
- Enter the text shown, with captions, **No. payments** etc.
- Place the cursor to the right of the caption, **No. payments** and press [ctrl][M] to insert a **Maths Box**.
- In the **Maths Box**, enter ***n:=tvm.n*** by pressing [var] to select ***tvm.n***, and [ctrl] [templates] (:=) for the **assign** command.
- Similarly, insert **Maths Boxes** next to the other captions and enter the assigned **tvm** functions, as shown.



To create an amortisation table, add a **Notes** page, then:

- Press [ctrl][M] to insert a **Maths Box**.
- Press [catalog][A] and navigate to **amortTbl** with syntax: amortTbl(*NPmt*, *N*, *I*, *PV*, [*Pmt*], [*FV*], [*PpY*], [*CpY*], [*PmtAt*], [*roundValue*]).
- Enter **amortTbl(*n*,*n*,*r*, *pv*,*pay*,*fv*,*ppy*,*cpy*)**, as shown.

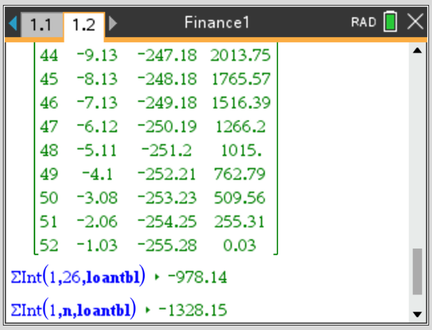
Note: ***1***. *Accept the defaults for the last two inputs:* ***PmtAt*** *(default is END of payment period),* ***roundValue*** *(default is 2).*
2. *Table columns: Payment period, interest paid, principal repayment, loan balance.*
3. ***NPmt*** *can be set to less than* ***N*** *to show part of the table.*
4. ***amortTbl*** *can also be selected from the Finance menu.*



(c) To determine the interest paid, on the **Notes** page:

- Assign a varible **loantbl** to the table by editing the **Maths Box** to **loantbl:= amortTbl(*n*,*n*,*r*, *pv*,*pay*,*fv*,*ppy*,*cpy*)**.
- Press [ctrl][M] to insert a **Maths Box**.
- Press [menu] > **Calculation** > **Finance** > **Amortisation** > **Interest Paid** and enter **ΣInt(1,12,*loantbl*)**.
- Repeat as above and enter **ΣInt(1,*n*,*loantbl*)**.

Note: The syntax is ΣInt(NPmt1,NPmt2,amortTable).
Press [var] *to select* ***loantbl***.



Answer. Interest paid, 12-month: $978.14, 2 years: $1,328.15.

Converting an amortTbl() matrix to a Spreadsheet table and creating related graphs

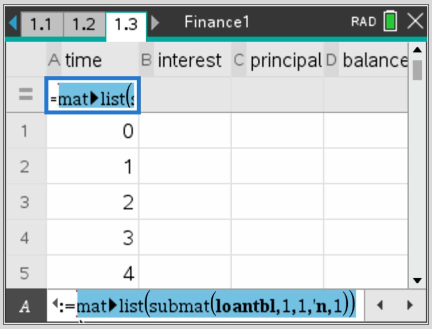
The previous problem relates to a $12,000 loan with an interest rate of 10.5% p.a., compounded fortnightly over a 2-year duration. The amortisation scheduled was created as a matrix.

(a) Transfer the values in the amortisation matrix to the Lists & Spreadsheet application.
(b) Use the values in the Lists & Spreadsheet page to graph the loan balance, cumulative payments made, and cumulative interest paid over the life of the loan.

(a) To convert the amortisation matrix to lists, add a **Lists & Spreadsheet** page to the previous problem, then:

- Enter the headings, **time**, **interest**, etc., as shown.
- In the column A formula cell, enter the formula **=mat▶list(subMat(*loantbl*,1,1,'*n*+1,1))**, by pressing [book] [1] [M] to select **mat▶list**, then [book] [S] to select **submat**, and [var] to select the stored variables, ***loantbl*** and ***n***.

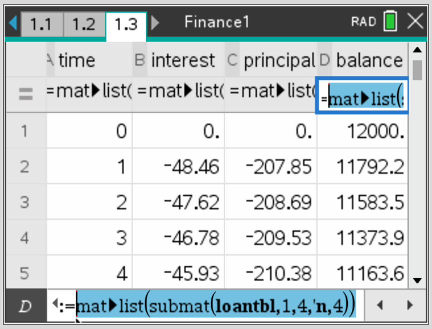
Note: The formula extracts column 1 from the matrix, with syntax: ***subMat****(Matrix,startRow,startCol,endRow, endCol).*



To extract columns 2 to 4 of the matrix, in the columns B, C and D formula cells, respectively, enter the formulas:

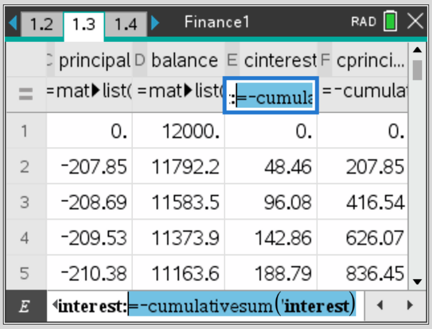
- **=mat▶list(subMat(*loantbl*,1,2,'*n*+1,2))** (for column B).
- **=mat▶list(subMat(*loantbl*,1,3,'*n*+1,3))** (for column C).
- **=mat▶list(subMat(*loantbl*,1,4,'*n*+1,4))** (for column D).

Note: To enter the above formulas, copy the column A formula by holding [shift] *and press* ◀ *to select the text, then press* [ctrl] [C] *to copy. Navigate to the next formula cell and press* [ctrl] [V] *to paste and edit the formula as needed.*



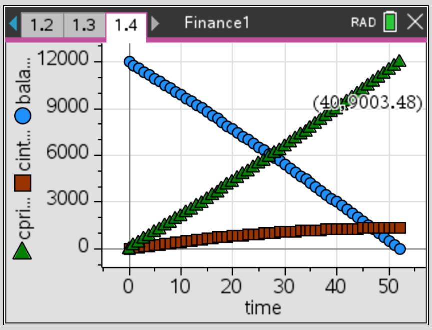
(b) To calculate the cumulative interest and cumulative principal payments made over the duration of the loan:

- In columns E and F, enter the headings, **cinterest** and **cprincipal**, as shown.
- In the column E formula cell, enter the formula **=cumulativeSum('interest)**, pressing [book] [D] to navigate to **cumulativeSum** and [var] to select **interest**.
- Similarly, in the column F formula cell, enter the formula **=cumulativeSum('principal)**.



To plot the loan balance, cumulative interest and cumulative principal payments against time (fortnightly repayments):

- Add a **Data & Statistics** page to the document.
- Press [tab] and select **time** for the horizontal axis.
- Press [tab] and select **balance** for the vertical axis.
- Press [menu] > **Plot Properties** > **Add Y Variable** and select **cprincipal**.
- Press [menu] > **Plot Properties** > **Add Y Variable** and select **cinterest**.



Note: The graphs could alternatively by created using Scatter Plot in the ***Graphs*** *application.*

3.5.4. Annuities and perpetuities

Analysing the value of an annuity after n payments on a step-by-step basis

Elia's investments include an annuity with a principal of \$100,000 to provide a monthly income of \$1200 per month until the annuity is exhausted. The annuity earns interest of 6.2% per annum, compounded monthly. The annuity is modelled by the recurrence relation

$u_0 = 100000,\ u_{n+1} = u_n \times \left(1 + \frac{6.2}{100 \times 12}\right) - 1200$, where u_n is the value of the annuity after n payments.

(a) In the Calculator application, generate a sequence to find the value of the annuity after the first 9 payments.

(b) Create a table to display the payment made, the interest earned, the reduction in the principal and the balance of the annuity for the first 24 months. Hence find the cumulative interest earned in the first 12 months and in the first 24 months.

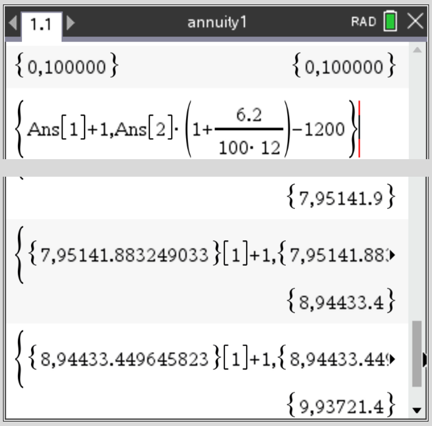
(c) On the same set of axes, create a graphical display of the value of the annuity, the cumulative reduction in the principal, and the cumulative interest earned over the first 24 months.

(d) State the value of the annuity, the cumulative interest earned and the cumulative reduction in the principal at the end of the first 24 months.

(a) To generate both the value of the annuity, u_n, and the corresponding payment period, n, on a **Calculator** page:

- Enter **{0,100000}**, being a list of {month,annuity value}.
- Enter $\left\{\mathbf{Ans}[1]+1, \mathbf{Ans}[2]\times\left(1+\frac{6.2}{100\times12}\right)-1200\right\}$, pressing ctrl (–) (ans) to key in **Ans**.
- Press enter to generate each new term of the sequence.

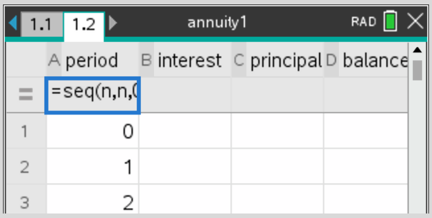
Answer: The output $\{n, u_n\} = \{9, 93721.4\}$ indicates that the value of the annuity after 9 payments is \$93,721.40.



(b) To set up a table to analyse the annuity:

- Press ctrl doc▾ (+page) > **Add Lists & Spreadsheet**.
- In the columns A, B, C and D heading cells, enter the headings as shown.

In the column A formula cell, enter **=seq(*n*,*n*,0,24)**.



To model the value of the annuity with a recurrence relation of $u_{n+1} = (1 + 6.2/(100\times12))\times u_n - 1200$:

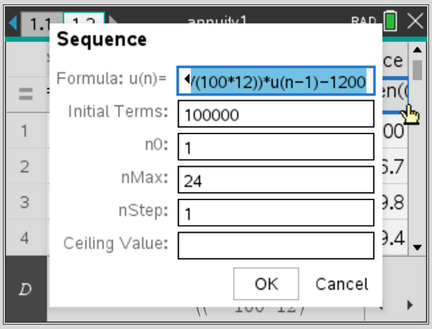
- Navigate to the column D formula cell (second from the top) and press menu > **Data** > **Generate Sequence**.

In the dialog box that follows, enter the inputs as shown, with Formula: $\boldsymbol{u(n) = (1+6.2/(100\times12))\times u(n-1)-1200}$

Note: Alternative to using the Generate Sequence command: Enter 100000 in cell D1, then in cell D2 enter the formula $\boldsymbol{=(1+6.2/(100\times12))\times d1-1200}$.

Navigate to cell D2, then press ctrl menu *> Fill.*

Press ▼ *to fill down to cell D25, then press* enter.



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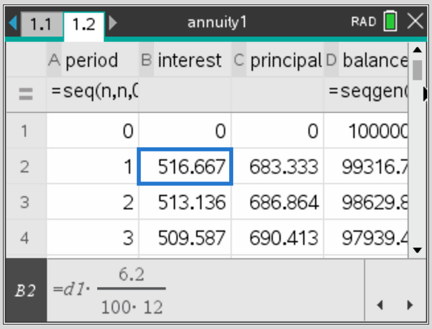
Analysing the value of an annuity after *n* payments on a step-by-step basis (continued)

To calculate the monthly interest earned over 24 months:

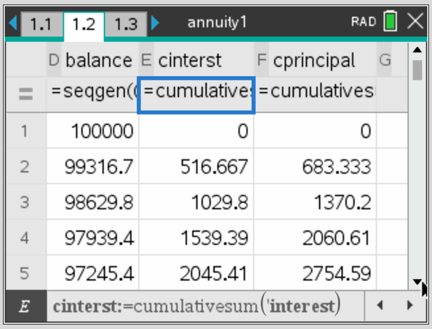


- Enter **0** in cell B1, then in cell B2 enter the formula $= \boldsymbol{d1} \times \left(\mathbf{6.2}/(\mathbf{100}\times\mathbf{12})\right)$.
- Navigate to cell B2, then press ctrl menu > **Fill**.
- Press ▼ to fill down to cell B25, then press enter.

To calculate the monthly reduction in the principal amount:

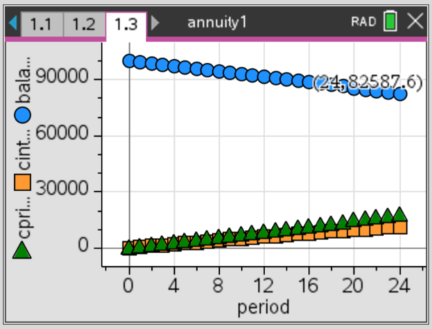
- Enter **0** in cell C1, then in cell C2 enter, **=1200 –** ***b*2**.
- Navigate to cell C2, then press ctrl menu > **Fill**.
- Press ▼ to fill down to cell C25, then press enter.

(c) To calculate the cumulative interest over the 24 months:



- In the columns E and F heading cells, enter the headings **cinterest** and **cprincipal**, as shown.
- In the columns E and F formula cells, enter the formulas **=cumulativeSum('interest)** and **=cumulativeSum('principal)** by pressing 📖 D to select **cumulativeSum**, and pressing var to select the stored lists, **interest** and **principal**.

To plot the annuity balance, cumulative interest earned and cumulative reduction in the principal against time:



- Add a **Data & Statistics** page to the document.
- Press tab and select **period** for the horizontal axis.
- Press tab and select **balance** for the vertical axis.
- Press menu > **Plot Properties** > **Add Y Variable** and select **cinterest**.
- Press menu > **Plot Properties** > **Add Y Variable** and select **cprincipal**.

(d) Answer. At the end of 24 months, the value of the annuity is $82,587.62, the cumulative interest earned is $11,387.62 and the cumulative reduction in the principal is $17,412.38.

Using financial modelling functionality to solve problems involving annuity investments

Assume that an annuity earns interest of 5.93% p.a., compounded quarterly.

(a) Determine the amount that needs to be invested, correct to the nearest dollar, to provide a quarterly income of $6000 for 10 years.

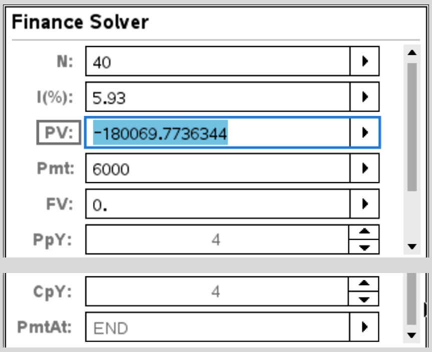
(b) Use the 'amortTbl' command to create an amortisation table for this annuity.

(c) Determine the minimum number of quarterly payment periods required for the value of the annuity to be less than half of the initial investment amount.

(d) The investor negotiates investing $180,000 to provide income of $6,000 per quarter for 10.5 years. Determine the interest rate, which compounds quarterly, correct to 2 decimal places.

(a) To determine the fortnightly loan repayment amount, add a **Notes** page to a new document, then:

- Press [menu] > **Calculations** > **Finance** > **Finance Solver**.
- On the dialog box, key in the values shown, except for the **PV** field. Press [tab] to navigate between fields.
- When all other fields have values allocated, navigate to the **PV** field and press [enter] to calculate the value.
- Press [var] to confirm that the values for each field have been stored as ***tvm*** variables, ***tvm.n***, ***tvm.i***, ***tvm.pv*** etc.

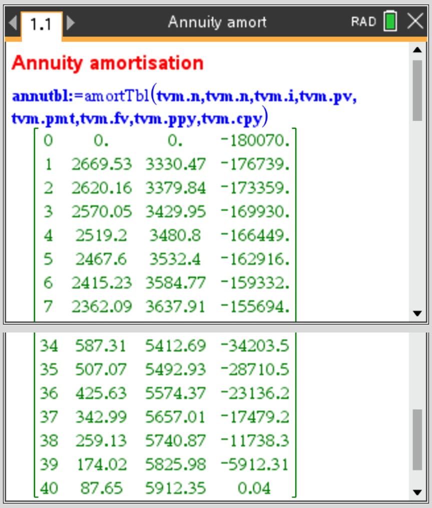


Answer. The required investment is $180,070 (correct to the nearest dollar).

(b) To create an amortisation table, on the **Notes** page:

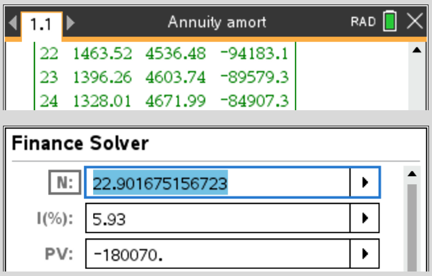
- Press [ctrl] [M] to insert a **Maths Box**.
- Press [📖] [1] [A] and navigate to **amortTbl** with syntax: amortTbl(*NPmt*, *N*, *I*, *PV*, [*Pmt*], [*FV*], [*PpY*], [*CpY*], [*PmtAt*], [*roundValue*]).
- Enter **annutbl:=amortTbl(*tvm.n*,tvm.*n*,tvm.*i*, tvm.*pv*, tvm.*pmt*, tvm.*fv*,tvm.*ppy*,tvm.*cpy*)**, by pressing [ctrl] [⊞] (:=) for the **assign** command and pressing [var] to selecting the **tvm** values.

Note: 1. Accept the defaults for the last two inputs: PmtAt (default is END of payment period), roundValue (default is 2).
2. Table columns: Payment period, interest earned, principal reduction, annuity balance.



(c) To determine when the value of the annuity is halved:

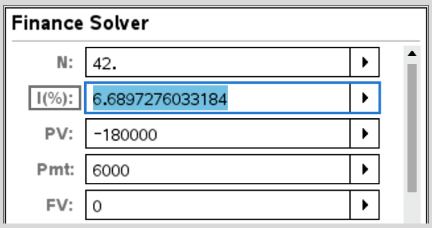
- Read the amortisation table (balance < $90,035)
- Press [menu] > **Calculation** > **Finance** > **Finance Solver**.
- On the dialog box, edit **FV** field to **90035**.
- Press [tab] to navigate to the **N** field, then press [enter].



Answer. The value of the annuity is less than half its original value after 23 payments.

(d) To determine the interest rate, on **Finance Solver**:

- Edit the fields, FV: **0**, N: $\mathbf{10.5 \times 4}$, PV: **180000**
- Press [tab] to navigate to the **I(%)** field, then press [enter].



Answer. The negotiated interest rate is 6.69% p.a.

Graphing the amortisation of an annuity

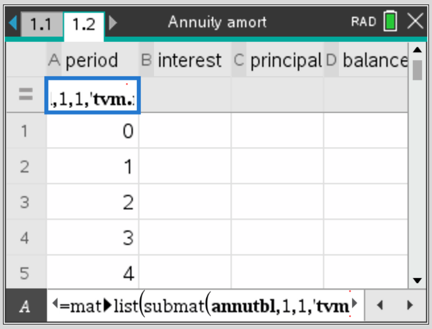
In the previous problem, $180,000 is invested in an annuity to provide income of $6,000 per quarter for 10.5 years. The interest rate is 6.69% p.a., compounded quarterly.

(a) Convert the amortisation matrix created above to lists in the Lists & Spreadsheet application.

(b) Use the amortisation data to plot the annuity balance, cumulative interest earned and cumulative reduction in the principal against the payment period.

(a) To convert the amortisation matrix to lists, add a **Lists & Spreadsheet** page to the previous problem, then:

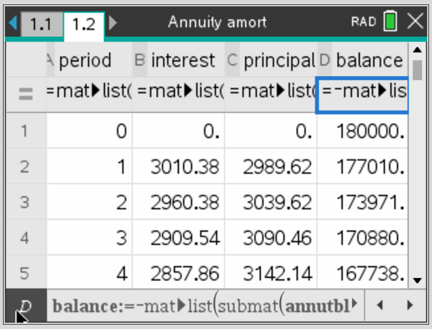
- Enter the headings, **period**, **interest**, etc., as shown.
- In the column A formula cell, enter the formula **=mat▶list(subMat(*annutbl*,1,1,'*tvm.n*+1,1))**, by pressing [📖][M] to select **mat▶list**, then [📖][S] to select **submat**, and [var] to select the variables, ***annutbl*** and **tvm.*n***.



Note: The formula extracts column 1 from the matrix, with syntax: ***subMat****(Matrix,startRow,startCol,endRow, endCol).*

To extract columns 2 to 4 of the matrix, in the columns B, C and D formula cells, respectively, enter the formulas:

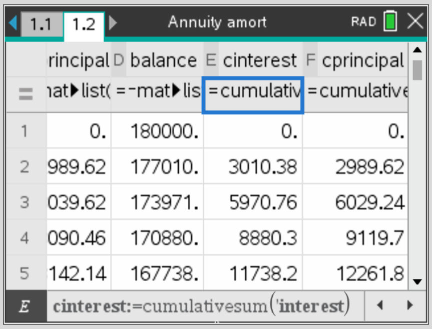
- **=mat▶list(subMat(*annutbl*,1,2,'tvm.*n*+1,2))**.
- **=mat▶list(subMat(*annutbl*,1,3,'tvm.*n* +1,3))**.
- **= –mat▶list(subMat(*annutbl*,1,4,'tvm.*n* +1,4))**.



Note: *To enter the above formulas, copy the column A formula by holding [⇧shift] and pressing ◀ to select the text, then pressing [ctrl][C] to copy. Navigate to the next formula cell and press [ctrl][V] to paste the formula, then edit as required.*

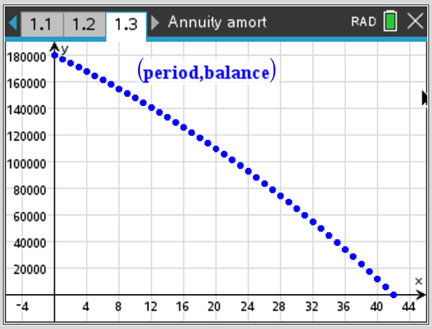
To calculate the cumulative interest earned and cumulative reduction in the principal over the duration of the annuity:

- In columns E and F, enter the headings, **cinterest** and **cprincipal**, as shown.
- In the column E formula cell, enter the formula **=cumulativeSum('interest)**, pressing [📖][D] to navigate to **cumulativeSum** and [var] to select **interest**.
- Similarly, in the column F formula cell, enter the formula **=cumulativeSum('principal)**.



(b) To plot the annuity balance against time:

- Add a **Graphs** page to the problem by pressing [ctrl][doc▾]([+page]) > **Add Graphs**.
- Press [menu] > **Graphs Entry/Edit** > **Scatter Plot**.
- For **S1**, enter ***x*** variable, pressing [var] then select **period**.
- Enter the ***y*** variable, pressing [var] to select **balance**.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: **–6**, XMax: **46**, XScale: **4**
  YMin: **–20000**, YMax: **180000**, YScale: **20000**



- Press [ctrl] [menu] > **Hide/Show** > **Show Lined Grid**.

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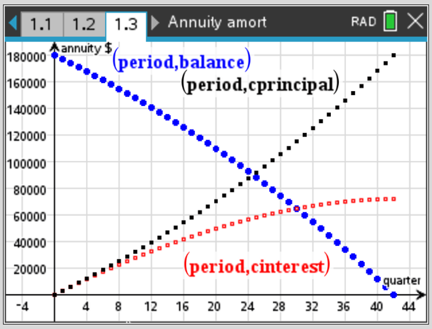
Graphing the amortisation of an annuity (continued)

To plot the graph of the cumulative interest earned and the cumulative reduction in the principal, on the **Graphs** page:

- Press [tab] to open the graph entry field.
- For **S2**, enter the ***x*** variable, pressing [var] to select **period**.
- To enter the ***y*** variable, press [var] then select **cinterest**.
- For **S3**, enter the ***x*** variable, pressing [var] to select **period**.
- To enter the ***y*** variable, press [var] then select **cprincipal**.
- To show multiple axes labels, hover over an axis and press [ctrl] [menu] > **Attributes**. Select **Multiple Labels**.
- Rename the axes labels. Click ([click]) the label ***x*** and edit it to **quarter**. Similarly, edit the label ***y*** to **annuity $**.



3.5.5. Compound interest investment with additions

Modelling an investment with periodic contributions to the principal: step-by-step approach

A bank offers a *Transaction Saver* account, with no account fees but pays zero interest. It also offers a *Growth Saver* account, which pays interest of 1% per month, credited monthly on the account balance at the end of the month.

(a) A customer plans to deposit an initial amount of $6,000 and then deposit $400 every month thereafter. Compare the account balances over 36 months for the two different savings accounts and determine the difference after 36 months.

(b) Verify graphically that the explicit rule for the *Transaction* and *Growth Saver* account balance after n months are given by $t_n = 6000 + 400n$ and $g_n = 46000 \times (1.01)^n - 40000$, respectively.

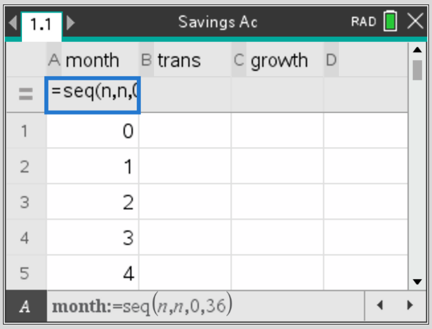
(a) To set up the spreadsheet, on a **Lists & Spreadsheet** page:

- In the column A, B and C heading cells respectively, enter the headings **month**, **trans** and **growth**, as shown.

To generate a list of 0 to 36 months, using a single formula:

- Navigate to the column A formula cell (second from the top) and enter the formula, **=seq(*n*,*n*,0,36)**.

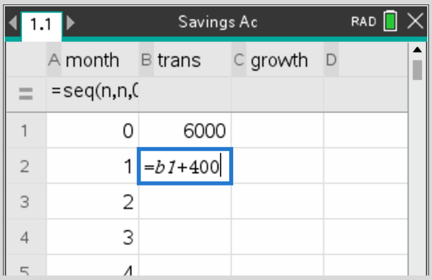
Note: *For '**seq**', press* [📖] [1] [S] *and navigate to **seq**. The syntax is **seq**(Expr, Var, Low, High [,Step]). Default Step = 1.*



To recursively generate the sequence of *Transaction* balances:

- In cell B1 enter 6000.
- In cell B2 enter the formula, **=*b*1+400**.
- Navigate to cell B2, press [ctrl] [menu] > **Fill**. Press ▼ to cell B37 then [enter]. This fills the formula down to cell B37.

Note: *The cell reference **b1** is relative to the cell location. When filled down, it renews to '**=b2+400**', '**=b3+400'**, etc.*



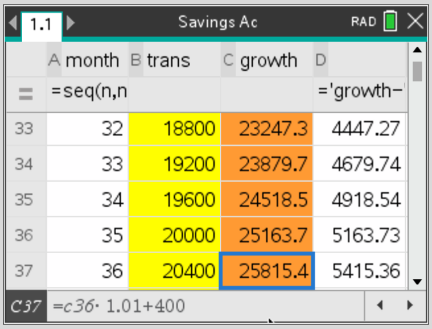
To recursively generate the sequence of *Growth* balances:

- In cell C1 enter 6000.
- In cell C2 enter the formula, **=*c*1×1.01+400**.
- Navigate to cell C2, press [ctrl] [menu] > **Fill**. Press ▼ to cell C37 then [enter]. This fills the formula down to cell C37.

Note: *The cell reference **c1** is relative to the cell location. When filled down, it renews to '**=c2×1.01+400**' etc.*

To calculate the account balance differences:

- In column D formula cell enter the formula, **='*growth*−'*trans*** by pressing [var] to select the list variables, **growth** and **trans**.



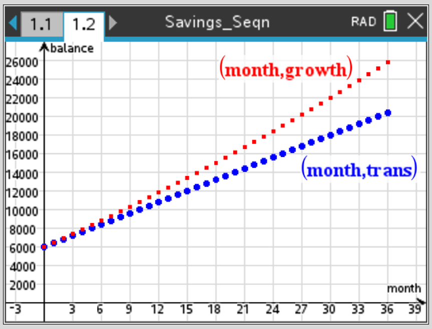
Answer. '*Growth*' is $5,415.36 greater after 36 months.

... continued

Modelling an investment with periodic contributions to the principal: step-by-step approach

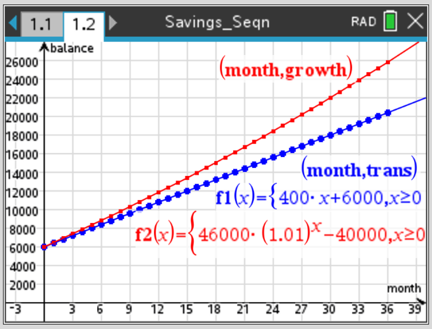
To show account balances as scatterplots, add a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Scatter Plot**.
- For **S1**, enter the ***x*** variable, press [var] then select **month**.
- To enter the ***y*** variable, press [var] then select **trans**.
- For **S2**, enter the ***x*** variable, press [var] then select **month**.
- To enter the ***y*** variable, press [var] then select **growth**.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: **–4**, XMax: **40**, XScale:**3**
  YMin: **–2000**, YMax: **28000**, YScale:**2000**
- Press [ctrl] [menu] > **Hide/Show** > **Show Lined Grid**.
- To show multiple axes labels, hover over an axis and press [ctrl] [menu] > **Attributes**. Select **Multiple Labels**.
- Rename the axes labels. Click (🖱) the label ***x*** and edit it to **month**. Similarly, edit the label ***y*** to **balance**.



(b) To show graphically that the explicit rules are $t_n = 6000 + 400n$ and $g_n = 46000 \times (1.01)^n - 40000$:

- Press [menu] > **Graph Entry/Edit** > **Function**.
- Enter $\boldsymbol{f1(x) = 400x + 6000 \mid x \geq 0}$
- Enter $\boldsymbol{f2(x) = 46000 \times (1.01)^x - 40000 \mid x \geq 0}$

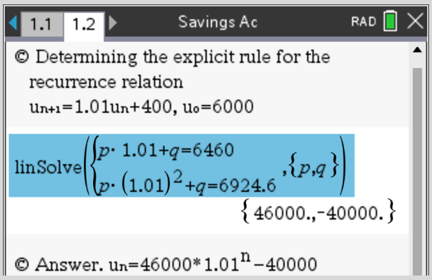


Answer. The explicit rules are confirmed by the correspondence of the function graphs and the scatterplots.

Note: *Although not required, the explicit rule for the Growth Saver account could be determined by solving the following pair of simultaneous equations for p, q:*

$g_1 = p(1.01)^1 + q = 6460$ *and* $g_2 = p(1.01)^2 + q = 6924.6$

The explicit rule $g_n = 46000(1.01)^n - 40000$ *is confirmed.*



Using financial modelling functionality to solve problems involving investments

Arya needs a deposit of $30,000 to qualify for a home loan. Araya deposits $12,000 in an investment account that earns interest of 8.7% per annum, compounded fortnightly.

(a) If Arya adds $350 per fortnight to the account, determine the account balance after 1 year.
(b) Create a table showing the fortnightly interest earned and account balance for the first year.
(c) If the interest rate is unchanged, verify that it will take Arya 43 months to qualify for the home loan.
(d) After one year, the interest rate decreases to 6.64% p.a. To qualify for the loan at the same time as previously, what fortnightly amount should Arya contribute from the 27th fortnight onwards?
(e) The balance of Arya's investment after *n* fortnightly time periods is modelled by the recurrence relation,

$$u_0 = 12000,\ u_{n+1} = \begin{cases} \left(1 + 8.7/(100 \times 26)\right) \times u_n + 350, & 0 \leq n \leq 26 \\ \left(1 + 6.64/(100 \times 26)\right) \times u_n + 370, & 27 \leq n \leq 43 \end{cases}$$

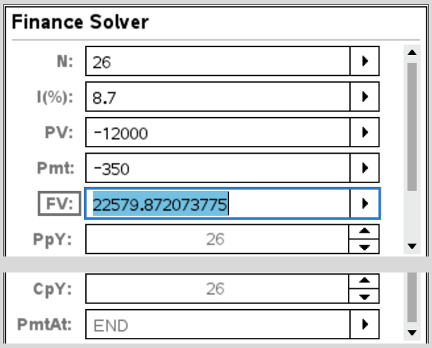
Create a tabular and a graphical display of the account balance after *n* time periods.

... continued

Using financial modelling functionality to solve problems involving investments (continued)

(a) To determine the account balance after 1 year if $r = 8.7\%$ p.a. and $350 is added fortnightly, on a **Calculator** page:

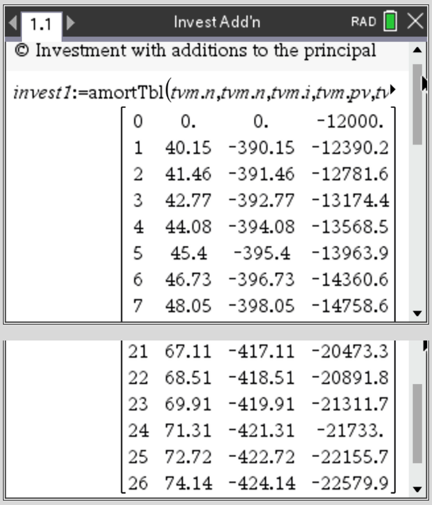
- Press [menu] > **Finance** > **Finance Solver**.
- In the dialog box, key in the values shown, except for the **FV** field. Press [tab] to navigate between fields.
- When all other fields have values allocated, navigate to the **FV** field and press [enter] to calculate the value.
- Press [var] to confirm that the values for each field have been stored as **tvm** variables, **tvm.*n***, **tvm.*i***, **tvm.*pv*** etc.



Answer. The balance is $22,579.87 after 1 year.

(b) To create a table showing the fortnightly interest earned and account balance for the first year:

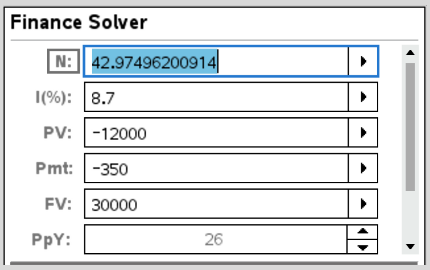
- Press [📖] [1] [A] and navigate to **amortTbl** with syntax: amortTbl(*NPmt*, *N*, *I*, *PV*, [*Pmt*], [*FV*], [*PpY*], [*CpY*], [*PmtAt*], [*roundValue*]).
- Enter **invest1:=amortTbl(*tvm.n*,tvm.*n*,tvm.*i*, tvm.*pv*, tvm.*pmt*, tvm.*fv*,tvm.*ppy*,tvm.*cpy*)**, by pressing [ctrl] [templates] ([:=]) for the **assign** command and pressing [var] to select the **tvm** variables.



Note:
1. Accept defaults for the last two inputs: PmtAt (default is END of payment period), roundValue (default is 2).
2. Table columns: Time period, interest earned, increase in the principal, and account balance.

(c) To determine how long it will it take Arya to qualify for the home loan if the interest rate is unchanged:

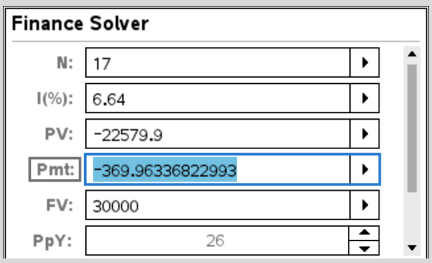
- Press [menu] > **Finance** > **Finance Solver**.
- Edit the field, FV: **30000**.
- Press [tab] to navigate to the **N** field, then press [enter].



Answer. Arya will reach the target amount of $30,000 after 43 fortnightly time periods.

(d) To determine fortnightly additions if $r = 6.64\%$ p.a. after the first year and the number of time periods remains at 43:

- Press [menu] > **Finance** > **Finance Solver**.
- Edit the fields, N: **43–26**, I(%): **6.64**, PV: **–22579.90**
- Press [tab] to navigate to the **Pmt** field, then press [enter].



Answer. To reach the target of $30,000 after 43 fortnightly time periods, Arya should deposit $370 each fortnight (to the nearest dollar).

... continued

Using financial modelling functionality to solve problems involving investments (continued)

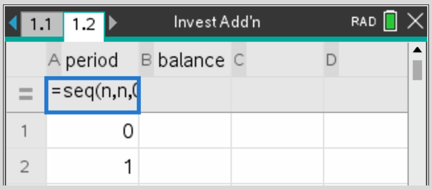
Approach 1: Tabular display

(e) To set up the table, on a **Lists & Spreadsheet** page:

- Enter the headings **period** and **balance**, as shown.

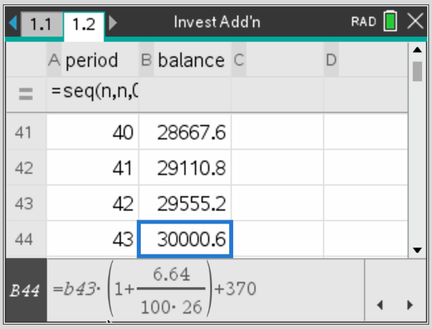
To generate a list of 0 to 43 fortnightly time periods:

- Navigate to the column **A** formula cell (second from the top) and enter the formula, **=seq(*n*,*n*,0,43)**.



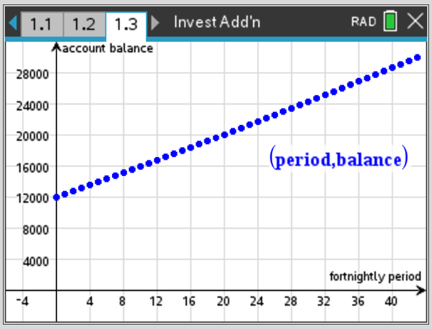
To recursively generate the sequence $u_0 = 12000,\ u_{n+1} = \ldots$:

- Enter **12000** in cell **B1**.
- In cell **B2**, enter the formula, **=*b*1×(1+8.7/(100×26))+350**.
- Navigate to cell **B2**, then press ctrl menu > **Fill**.
- Press ▾ to fill down to cell **B27**, then press enter.
- In cell **B28**, enter, **=*b*27×(1+6.64/(100×26))+370**.
- Navigate to cell **B28**, then press ctrl menu > **Fill**.
- Press ▾ to fill down to cell **B44**, then press enter.



To plot the sequence, add a **Graphs** page, then:

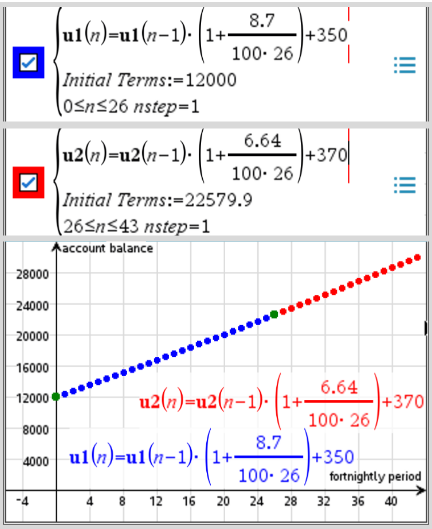
- Press menu > **Graph Entry/Edit** > **Scatter Plot**.
- For **S1**, enter the ***x*** variable, press var then select **period**.
- To enter the ***y*** variable, press var then select **balance**.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: **–6**, XMax: **44**, XScale:**4**
  YMin: **–4000**, YMax: **32000**, YScale:**4000**



Approach 2: Sequence graph display

To plot the sequence, on a **Graphs** page:

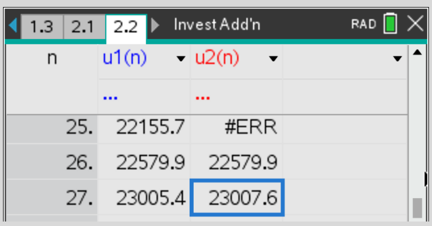
- Press menu > **Graph Entry/Edit** > **Sequence** > **Sequence**.
- Enter the inputs for ***u*1(*n*)** and ***u*2(*n*)**, as shown.
- Press menu > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: **–6**, XMax: **44**, XScale:**4**
  YMin: **–4000**, YMax: **32000**, YScale:**4000**
- Press ctrl menu > **Hide/Show** > **Show Lined Grid**.
- To show multiple axes labels, hover over an axis and press ctrl menu > **Attributes**. Select **Multiple Labels**.
- Rename the axes labels. Click ( ) the label ***x*** and edit it to **period**. Similarly, edit the label ***y*** to **balance**.



To obtain a table of values from the sequence plots:

- Press ctrl T to show the table in a split page.
- Press doc▾ > **Page Layout** > **Ungroup**.

The table of values appears on a separate page, with the values for ***u*2(*n*)** starting at $n = 26$.



VCE General Mathematics Unit 4

4.1. Matrices and their applications

Creating different types of matrices

The order of a matrix is $m \times n$, where *m* is the number of rows and *n* is the number of columns.

For the matrix *A*, which has *m* rows and *n* columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$ where a_{mn} is the element in row *m* and column *n*.

***Note**: Section 1.4.1 (page 34) demonstrates how to create row, column, square, zero and identity matrices with the TI-Nspire CX II CAS.*

A diagonal matrix is a square matrix whose elements outside the leading (main) diagonal (from top-left to bottom-right) are all zero.

The elements on the leading diagonal can be any value, including zero.

(a) Create the diagonal matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

A symmetric matrix is a square matrix that is unchanged by transposition (switching rows and columns).

In a symmetric matrix, the elements above the leading diagonal are a mirror image of the elements below the leading diagonal.

(b) Verify that the matrix $S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is symmetric.

A triangular matrix is a square matrix where either all elements above the leading diagonal are zero (lower triangular) or all elements below the leading diagonal are zero (upper triangular).

Diagonal matrices are a special case of triangular matrices.

(c) Create the triangular matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

A binary matrix is a matrix where each element is either a 0 or a 1.

The identity matrix, *I*, is an example of a binary matrix.

(d) Create the binary matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

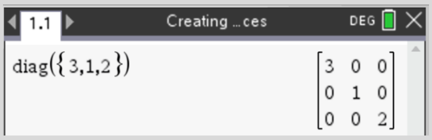
... continued

Creating different types of matrices (continued)

(a) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a 3×3 diagonal matrix with elements 3, 1 and 2 on the leading diagonal.

On a **Calculator** page, create the matrix as follows:

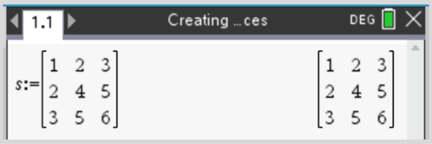
- Press menu > **Matrix & Vector** > **Create** > **Diagonal**.
- Enter as shown.



(b) $S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is a 3×3 matrix.

On a **Calculator** page, assign S as follows:

- Press ctrl [templates] to access the **Assign** := command.
- Press [catalog] 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 3.
- Enter as shown.



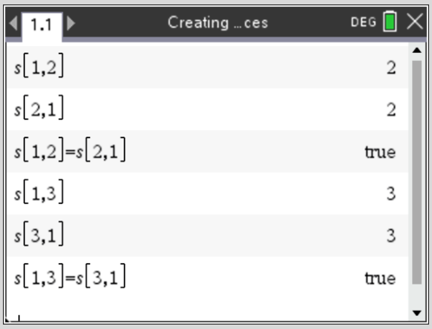
For S to be a symmetric matrix, we require $s_{12} = s_{21}, s_{13} = s_{31}$ and $s_{23} = s_{32}$.

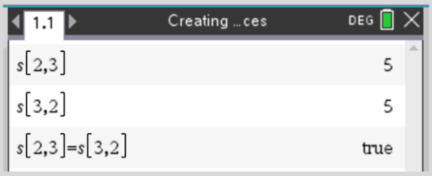
To access these elements, enter as shown.

$s_{12} = s_{21} = 2$, $s_{13} = s_{31} = 3$ and $s_{23} = s_{32} = 5$

Hence S is symmetric.

***Note:** The first element in $[1, 2]$ indicates the row number and the second element indicates the column number.*



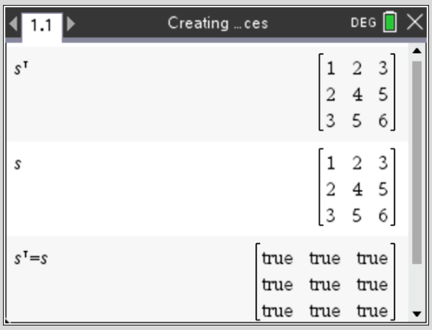


In essence, the transpose of matrix S is equal to matrix S i.e. $S^T = S$. This is shown as follows:

On a **Calculator** page, create the transpose of S as follows:

- Press menu > **Matrix & Vector** > **Transpose**.

Notes:
*1. The superscript **T** can also be accessed from the symbols palette (ctrl [catalog]) or [catalog] T.*
2. See Page 126 for another example showing how to generate the transpose of a matrix.



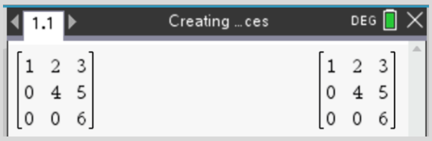
... continued

Creating different types of matrices (continued)

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is a 3×3 (upper) triangular matrix.

On a **Calculator** page, create the matrix as follows:

- Press 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 3.
- Enter as shown.

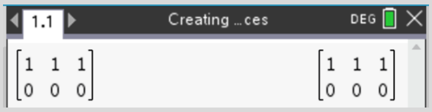


(d) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is a 2×3 binary matrix.

On a **Calculator** page, create the matrix as follows:

- Press 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 2 and the number of columns to be 3.

Enter as shown.



Constructing a matrix given a rule for the a_{ij} element

Matrices can be constructed using a rule that connects each element with its row and column number and its order.

A is a 1×3 matrix. The element in row i and column j of A is given by $a_{ij} = i + j^2$.

Construct matrix A.

Use the rule $a_{ij} = i + j^2$ to generate the elements of

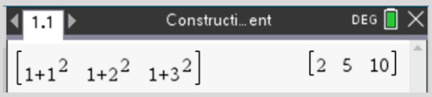
$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$.

Substituting for i and j into $a_{ij} = i + j^2$ gives

$A = \begin{bmatrix} 1+1^2 & 1+2^2 & 1+3^2 \end{bmatrix}$.

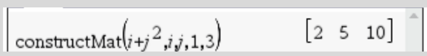
On a **Calculator** page, create A as follows:

- Press 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 1 and the number of columns to be 3.
- Enter as shown.



So $A = \begin{bmatrix} 2 & 5 & 10 \end{bmatrix}$.

Note**: The TI-Nspire CX II CAS also has a command that will construct a matrix based on a given rule. The command **constructMat(Expr,Var1,Var2,numRows,numCols)**. The command can be found via* menu *> **Matrix & Vector** > **Create** > **Construct Matrix**. Enter **constructMat(i+j²,i,j,1,3) as shown.



Introducing the transpose of a matrix

The transpose of a matrix involves switching its rows and columns, effectively flipping it over its leading diagonal.

Given a matrix A, the transpose of A is denoted by A^T.

The transpose of a row matrix is a column matrix and vice versa.

As the rows and columns are switched, the transpose of an $m \times n$ matrix is an $n \times m$ matrix.

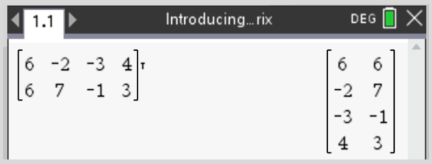
Determine the transpose of the matrix $\begin{bmatrix} 6 & -2 & -3 & 4 \\ 6 & 7 & -1 & 3 \end{bmatrix}$.

On a **Calculator** page, create the transpose of

$\begin{bmatrix} 6 & -2 & -3 & 4 \\ 6 & 7 & -1 & 3 \end{bmatrix}$ as follows:

- Press [doc] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 2 and the number of columns to be 4.
- Enter as shown.
- Press [menu] > **Matrix & Vector** > **Transpose**.

$$\begin{bmatrix} 6 & -2 & -3 & 4 \\ 6 & 7 & -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 6 & 6 \\ -2 & 7 \\ -3 & -1 \\ 4 & 3 \end{bmatrix}$$



***Note:** The superscript **T** can also be accessed from the symbols palette ([ctrl] [doc]) or [doc] [T].*
This key is not the same as the keypad key [T].

Applying matrix arithmetic

Two matrices can be added or subtracted if they have the same order.

For any two matrices, A and B of the same order, $A + B = B + A$.

Matrices can be multiplied by scalar (real number) quantities.

Norah has a job observing and recording the route that vehicles take through a traffic intersection.

In her 1st hour on duty, Norah observed that 210 vehicles turned left (L), 186 turned right (R), 6 performed a U-turn (U) and 396 travelled straight through the intersection (S).

Greta has a similar job to Norah at the next intersection.

She observed that 171 vehicles turned left (L), 156 turned right (R), 3 performed a U-turn (U) and 360 travelled straight through the intersection (S).

(a) Show this information in a 2×4 matrix A, where A is of the form

$$\begin{matrix} & L \quad U \quad R \quad S \\ \text{Norah} & \begin{bmatrix} \quad & & & \quad \\ \quad & & & \quad \end{bmatrix} \\ \text{Greta} & \end{matrix}.$$

Applying matrix arithmetic (continued)

At the end of the 2nd hour, the matrix B, representing the total number of observed vehicles is as follows:

$$\begin{array}{r} \\ \text{Norah} \\ \text{Greta} \end{array}\begin{array}{c} \begin{array}{cccc} \text{L} & \text{U} & \text{R} & \text{S} \end{array} \\ \begin{bmatrix} 460 & 402 & 10 & 910 \\ 349 & 392 & 15 & 810 \end{bmatrix} \end{array}.$$

(b) Find $B - A$ and interpret the result.

(c) In the 3rd hour, Norah and Greta observed a number of vehicles given by $C = \frac{1}{3}A$.

(i) Find matrix C. **(ii)** Interpret c_{14} i.e. element $(1, 4)$.

(d) Using matrix methods, find a 2×4 matrix that represents all observed vehicle movements for the first three hours.

Past experience has suggested to Norah and Greta that during the 4th hour, there is close to a 10 per cent increase in all observed vehicle movements compared to the 3rd hour.

(e) Find a matrix that represents the 4th hour observation of vehicles. Give each element in the matrix correct to the nearest vehicle.

Norah and Greta finish observing vehicles after six hours.

At this time, they generate a matrix, shown below, which shows the hourly averages for each observed vehicle movement.

$$\begin{array}{r} \\ \text{Norah} \\ \text{Greta} \end{array}\begin{array}{c} \begin{array}{cccc} \text{L} & \text{U} & \text{R} & \text{S} \end{array} \\ \begin{bmatrix} 102 & 98 & 3 & 320 \\ 88 & 95 & 4 & 310 \end{bmatrix} \end{array}$$

(f) Use matrix methods to find the 6-hour observed vehicle totals in a 2×4 matrix.

(g) If the matrices representing the observed vehicles in the 5th and 6th hours are identical, determine each matrix. Give each element in the matrix correct to the nearest vehicle.

(h) How many observed vehicles turned right at Norah's intersection in the 5th hour?

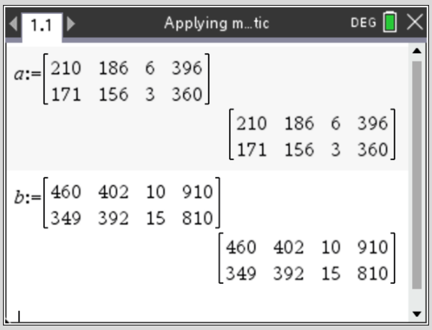
(a) Matrix A is $\begin{bmatrix} 210 & 186 & 6 & 396 \\ 171 & 156 & 3 & 360 \end{bmatrix}$.

(b) Matrix B is $\begin{bmatrix} 460 & 402 & 10 & 910 \\ 349 & 392 & 15 & 810 \end{bmatrix}$.

On a **Calculator** page, assign A and B as follows:

- Press ctrl [templates] to access the **Assign** := command.
- Press [book] 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 2 and the number of columns to be 4.
- Enter as shown.

__Note:__ The __Define__ command (menu > __Actions__ > __Define__) or the __Store__ command (press ctrl var to access sto→) can also be used.

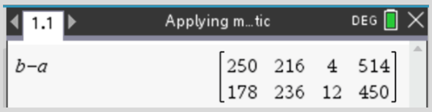


... continued

Applying matrix arithmetic (continued)

- Enter $B-A$ as shown.

$$B-A=\begin{bmatrix}460 & 402 & 10 & 910\\ 349 & 392 & 15 & 810\end{bmatrix}-\begin{bmatrix}210 & 186 & 6 & 396\\ 171 & 156 & 3 & 360\end{bmatrix}$$
$$=\begin{bmatrix}250 & 216 & 4 & 514\\ 178 & 236 & 12 & 450\end{bmatrix}$$



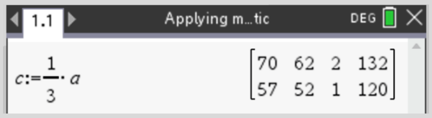
The matrix $B-A$ represents car movements during the 2nd hour.

(c) (i) In the 3rd hour, $C=\frac{1}{3}A$.

On a **Calculator** page, assign C as follows:

- Press ctrl [templates] to access the **Assign** [:=] command.
- Press ctrl [÷] to access the **Fraction** template.
- Enter as shown.

$$C=\frac{1}{3}\begin{bmatrix}210 & 186 & 6 & 396\\ 171 & 156 & 3 & 360\end{bmatrix}$$
$$=\begin{bmatrix}70 & 62 & 2 & 132\\ 57 & 52 & 1 & 120\end{bmatrix}$$



(ii) Element c_{14} is element $(1,4)$.

To access element $(1,4)$, enter as shown.

$c_{14}=132$

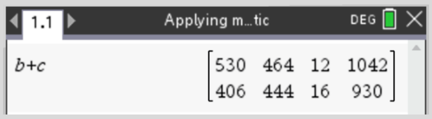
Norah observed 132 vehicles going straight through the intersection during the 3rd hour.

Note: *The first element in* $[1,4]$ *indicates the row number and the second element indicates the column number.*

(d) Since B gives the total number of observed vehicles by the end of the second hour and C gives the number observed during the third hour, the total number of observed movements over the first three hours is given by $B+C$.

Enter as shown.

$$B+C=\begin{bmatrix}460 & 402 & 10 & 910\\ 349 & 392 & 15 & 810\end{bmatrix}+\begin{bmatrix}70 & 62 & 2 & 132\\ 57 & 52 & 1 & 120\end{bmatrix}$$
$$=\begin{bmatrix}530 & 464 & 12 & 1042\\ 406 & 444 & 16 & 930\end{bmatrix}$$



...continued

Applying matrix arithmetic (continued)

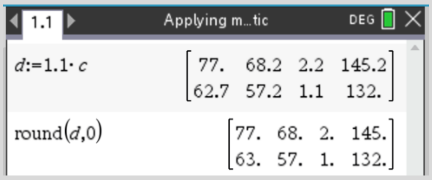
(e) To calculate the result of a (close to) 10% increase, we can multiply matrix C by 1.1.

Let matrix D represent the number of observed vehicles in each category during the 4th hour where $D=1.1C$.

On a **Calculator** page, assign D as follows:

- Press ctrl to access the **Assign** := command.
- Enter as shown.
- Press menu > **Number** > **Number Tools** > **Round** to give each element in the matrix correct to the nearest vehicle.

$$D=1.1\begin{bmatrix}70 & 62 & 2 & 132\\57 & 52 & 1 & 120\end{bmatrix}$$
$$=\begin{bmatrix}77 & 68 & 2 & 145\\63 & 57 & 1 & 132\end{bmatrix}$$



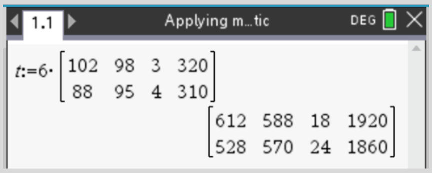
***Note**: The general syntax for the **Round** command is **round(Value [,Digits])**.*

(f) The 6-hour observed vehicle totals can be found by multiplying the averages by 6.

On a **Calculator** page, assign T as follows:

- Press ctrl to access the **Assign** := command.
- Press 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 2 and the number of columns to be 4.
- Enter as shown.

$$T=6\begin{bmatrix}102 & 98 & 3 & 320\\88 & 95 & 4 & 310\end{bmatrix}=\begin{bmatrix}612 & 588 & 18 & 1920\\528 & 570 & 24 & 1860\end{bmatrix}$$



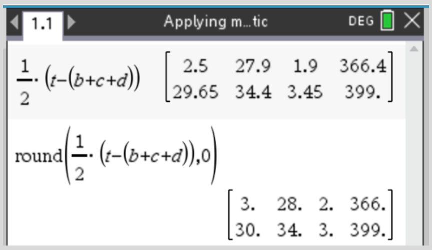
(g) Let E and F, where $E=F$, be the matrices representing the observed vehicles in the 5th and 6th hours respectively.

$$T=B+C+D+E+F$$
$$=B+C+D+2E$$

So $E=\frac{1}{2}\left(T-\left(B+C+D\right)\right)$.

- Enter as shown.
- Press menu > **Number** > **Number Tools** > **Round** to give each element in the matrix correct to the nearest vehicle.

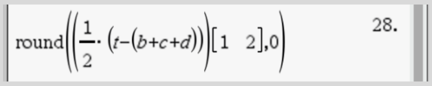
$$E=\begin{bmatrix}3 & 28 & 2 & 366\\30 & 34 & 3 & 399\end{bmatrix}$$



...continued

Applying matrix arithmetic (continued)

(h) The number of observed vehicles that turned right at Norah's intersection in the 5th hour is given by element $(1,2)$ in matrix E.



- Enter as shown.
- Press [menu] > **Number** > **Number Tools** > **Round** to give the element in the matrix correct to the nearest vehicle.

$e_{12} = 28$

Norah observed 28 vehicles turning right in the 5th hour.

***Note**: The general syntax for the **Round** command is **round(Value [,Digits])**.*

Using matrix products

Two matrices can be multiplied together if the number of columns in the first matrix equals the number of rows in the second matrix.

In general, if an $m \times n$ matrix is multiplied by an $n \times p$ matrix, the resulting matrix will be of order $m \times p$, that is, $(m \times \boldsymbol{n}) \times (\boldsymbol{n} \times p) = m \times p$.

In general, for two matrices, A and B, $AB \neq BA$, matrix multiplication is not commutative.

Where matrix multiplication is permitted, $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$.

In Australian rules football, a goal is worth 6 points and a behind is worth 1 point.

Geelong and Port Adelaide played each other in the 2007 AFL Grand Final.

At the end of the game:

Geelong had scored 24 goals 19 behinds.

Port Adelaide had scored 6 goals 8 behinds.

(a) Use matrix multiplication to find the number of points scored by Geelong and Port Adelaide.

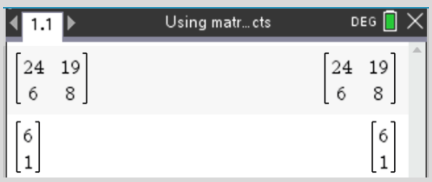
(b) Determine Geelong's winning margin in the game.

(a) Represent both team's goals and behinds in a 2×2 matrix.

On a **Calculator** page, create this matrix as follows:

- Press [📖] [5] and select the **2-by-2 Matrix** template.
- Enter as shown.

Represent the point values for a goal and a behind in a 2×1 matrix.



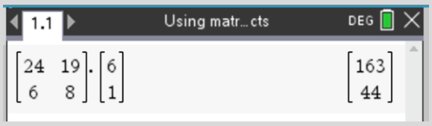
- Press [📖] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 2 and the number of columns to be 1.
- Enter as shown.

... continued

Using matrix products (continued)

If a 2×2 matrix is multiplied by an 2×1 matrix, the resulting matrix will be of order 2×1.

$$\begin{bmatrix}24 & 19\\ 6 & 8\end{bmatrix}\begin{bmatrix}6\\ 1\end{bmatrix}=\begin{bmatrix}163\\ 44\end{bmatrix}$$

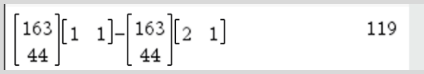


Geelong's score was 24 goals 19 behinds 163 points.

Port Adelaide's score was 6 goals 8 behinds 44 points

Notes:
1. *When performing matrix multiplication, always use the multiplication key,* [×].
2. *When attempting to multiply two matrices of different order, a '**dimension error**' message is displayed when the number of columns in the first matrix does not equal the number of rows in the second matrix.*

(b) To determine Geelong's winning margin, enter as shown.



The winning margin was 119 points.

Note: *The first element in* $\begin{bmatrix}1 & 1\end{bmatrix}$ *indicates the row number and the second element indicates the column number. The expression shown right is subtracting the element in the second row from the element in the first row of the matrix.*

Raising a matrix to a power

Consider $M=\begin{bmatrix}-2 & 9\\ -1 & 4\end{bmatrix}$.

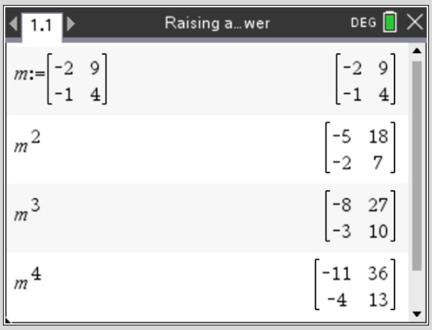
(a) Find M^2, M^3 and M^4.

(b) Hence, infer a general result for M^n, where n is a positive integer.

(c) Use your result to determine M^{50} and check your answer with *TI-Nspire CX II CAS*.

On a **Calculator** page, assign M as follows:

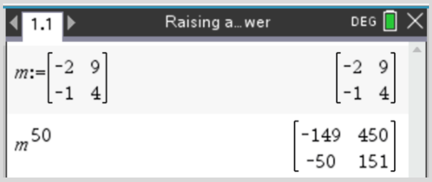
- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [book] [5], select the **2-by-2 Matrix** template and enter as shown.



(a) $M^2=\begin{bmatrix}-5 & 18\\ -2 & 7\end{bmatrix}$, $M^3=\begin{bmatrix}-8 & 27\\ -3 & 10\end{bmatrix}$ and $M^4=\begin{bmatrix}-11 & 36\\ -4 & 13\end{bmatrix}$.

(b) $M^n=\begin{bmatrix}1-3n & 9n\\ -n & 1+3n\end{bmatrix}$.

(c) $M^{50}=\begin{bmatrix}1-3(50) & 9(50)\\ -50 & 1+3(50)\end{bmatrix}=\begin{bmatrix}-149 & 450\\ -50 & 151\end{bmatrix}$.



Note: *Only a square matrix can be raised to a power.*

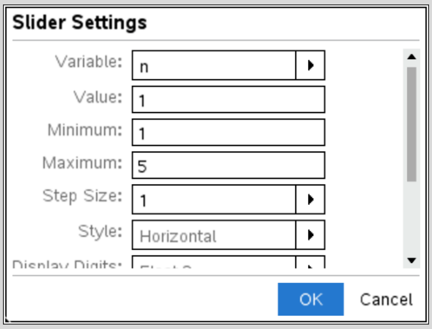
... continued

Raising a matrix to a power (continued)

Alternatively, on a **Notes** page:

Insert a **Slider** to control the value of n as follows:

- Press [menu] > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.



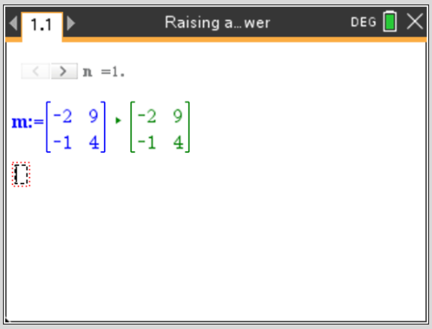
Insert a **Maths Box** as follows:

- Press [menu] > **Insert** > **Maths Box**.

Note: *Alternatively, to insert a* ***Maths Box****, press* [ctrl] [M].

Assign M as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **2-by-2 Matrix** template and enter as shown.

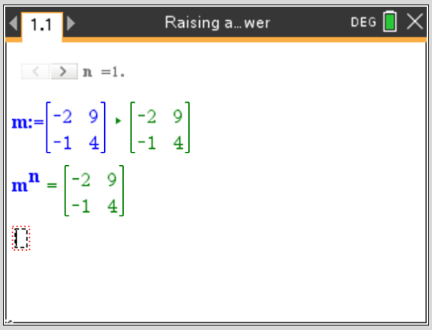


Now:

- Insert another **Maths Box** and enter m^n.

To change the display of a **Maths Box**, for example, to display an equals sign:

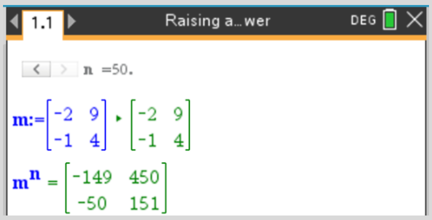
- Click on the **Maths Box**.
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▸ and select $=$.



Note: Maths Box Attributes *can also be accessed within a* ***Maths Box*** *by pressing* [ctrl] [menu].

Click on the slider to change the value of n.

To calculate M^n for $n = 50$, manually change the value of n to 50 in the slider box.



Verifying an inverse matrix property

Recall from Section 1.4.2 (page 44):

- $AA^{-1} = I = A^{-1}A$ where A^{-1} is the inverse of A and I is the identity matrix.

The inverse of a triangular matrix is a triangular matrix.

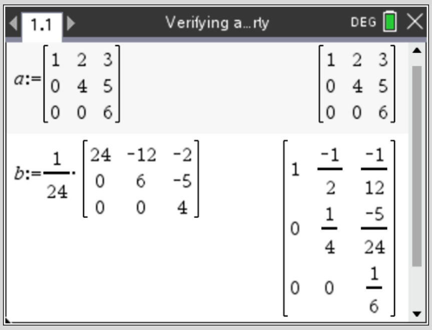
Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ and $B = \frac{1}{24}\begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}$.

Verify that $AB = I = BA$ so that $B = A^{-1}$.

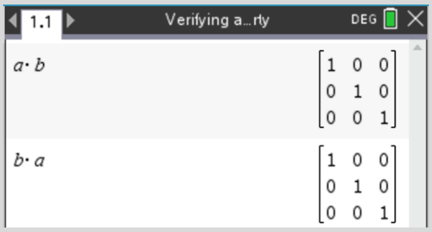
***Note**: The **Calculation Mode** must be set to either **Auto** or **Exact**. This can be done via [on] > Settings > Document Settings.*

On a **Calculator** page, assign A and B as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 3.
- Enter as shown.

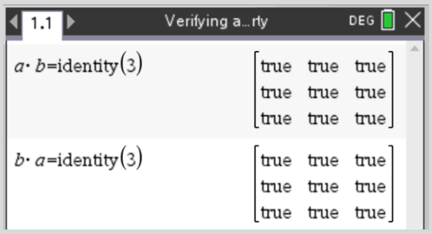


$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.



It can be concluded that $B = A^{-1}$.

Notes:
***(1)** Entering $AB = I$ and $BA = I$ both give the output*
$\begin{bmatrix} \text{true} & \text{true} & \text{true} \\ \text{true} & \text{true} & \text{true} \\ \text{true} & \text{true} & \text{true} \end{bmatrix}$.
***(2)** To enter matrix I, press [menu] **> Matrix & Vector > Create > Identity** and enter as shown.*



Determining whether a matrix is singular or non-singular

An $n \times n$ matrix A has an inverse if and only if its determinant $\det(A) \neq 0$.

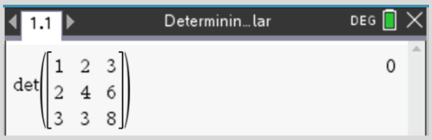
If $\det(A) = 0$, then A is a singular matrix and A^{-1} does not exist (i.e. A has no inverse).

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 3 & 8 \end{bmatrix}$.

Find $\det(A)$ and hence determine whether A is singular or non-singular.

On a **Calculator** page:

- Press [menu] > **Matrix & Vector** > **Determinant**.
- Press [📖] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 3.
- Enter as shown.



$\det(A) = 0$ and so A is singular.

***Note**: Since 2 is a common factor of the elements of the second row, the determinant can be expressed as*

$2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix}$. *The resulting determinant has two identical rows.*

If the corresponding elements of two rows (or columns) of a square matrix A are equal, the determinant is zero. When the two equal rows of the matrix are interchanged, the matrix remains the same and hence the value of their determinants are equal. However, the values of their determinants are equal in magnitude but opposite in sign. This occurs only when the determinant of the matrix is zero.

Solving matrix equations

Recall from Section 1.4.3 (page 46):

If $AX = B$, where A is a square matrix and has inverse A^{-1} such that $A^{-1}A = I$, then $X = A^{-1}B$.

Other matrix equations include $XA = B$ and $AX + BX = C$.

Matrices can be used to encode and decode messages.

In this coding method, assign each letter of the alphabet with its position number in the alphabet.

So $A = 1$, $B = 2, \ldots, Z = 26$.

For example, to send the message *GO CATS*, write the letters in a 2×4 matrix *M*.

$$M = \begin{bmatrix} & G & O & \\ C & A & T & S \end{bmatrix}$$

Replace each letter of the alphabet with its position number in the alphabet and use a zero to represent a space.

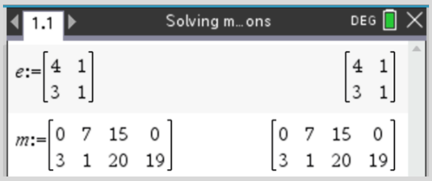
$$M = \begin{bmatrix} 0 & 7 & 15 & 0 \\ 3 & 1 & 20 & 19 \end{bmatrix}$$

This code is fairly easy to crack. However, by multiplying *M* by a suitably sized encoding matrix, *E*, this message can be made more difficult to decode.

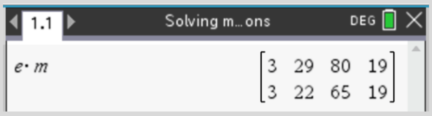
Let $E = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ and form the product *EM*.

On a **Calculator** page, assign *E* and *M* as follows:

- Press ctrl [template key] to access the **Assign** := command.
- For *E*, press [catalog] 5, select the **2-by-2 Matrix** template and enter as shown.
- For *M*, press [catalog] 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 2 and the number of columns to be 4.
- Enter as shown.



$$EM = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 7 & 15 & 0 \\ 3 & 1 & 20 & 19 \end{bmatrix} = \begin{bmatrix} 3 & 29 & 80 & 19 \\ 3 & 22 & 65 & 19 \end{bmatrix}$$

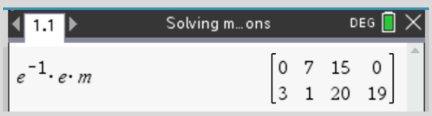


The encoded message sent to the recipient is the matrix product *EM*.

To decode this message, the recipient must pre-multiply the matrix product *EM* by E^{-1}.

$E^{-1}EM = M$ as $E^{-1}E = I$ and $IM = M$.

$$E^{-1}EM = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 3 & 29 & 80 & 19 \\ 3 & 22 & 65 & 19 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 7 & 15 & 0 \\ 3 & 1 & 20 & 19 \end{bmatrix}$$



Solving matrix equations (continued)

The recipient now replaces each letter's position number in the alphabet with the corresponding letter and inserts a space for the zero.

So $M = \begin{bmatrix} & G & O & \\ C & A & T & S \end{bmatrix}$ and the message received is *GO CATS*.

The trick to decoding messages of this type is to know E^{-1}, the inverse matrix of the encoding matrix *E*.

A good encoding matrix *E* is one that has $\det(E) = \pm 1$ as this avoids the use of fractions when decoding a message.

You are encouraged to encode and decode various messages using the above approach.

It is a good idea to vary *E*.

Applying matrix equations and inverses

Gemma sells watches in stores in Richmond, South Yarra and Geelong.

She has watches for men, women and children, each in three categories: *Basic*, *Standard* and *Special*.

The following table shows the yearly allocation of watches for each store.

	Richmond			South Yarra			Geelong		
	Women	Men	Children	Women	Men	Children	Women	Men	Children
Basic	90	40	50	40	20	30	80	70	60
Standard	100	80	23	120	110	32	80	60	15
Special	50	60	10	75	50	10	35	40	8

(a) For each store, write down a 3×3 matrix that represents the yearly watch allocation.

(b) The three matrices from part **(a)** can be added to form a matrix *A*.

(i) Determine matrix *A*.

(ii) Interpret the meaning of a_{23} i.e. element $(2, 3)$ in *A*.

Gemma creates a 1×3 matrix *C*, that shows the wholesale price she pays for each type of watch. Each *Basic* watch has a wholesale price of \$42, each *Standard* watch has a wholesale price of \$62 and each *Special* watch has a wholesale price of \$85.

(c) Find *CA* and interpret the result.

Gemma decides that she needs to make a 25% profit on all watches sold.

(d) Form a 1×3 matrix *S*, that shows the selling price of each type of watch.

Let *P* be a profit matrix.

(e) Use *A*, *S* and *C* to determine *P*.

... continued

Applying matrix equations and inverses (continued)

In December, Gemma opens a new store in the city and sells each of the three types of watches at a discounted price. She decides to allocate the same amount of stock that is allocated to the Geelong store annually. At the end of the sale, Gemma had sold all stock, and her takings were represented by the following matrix T.

$$\begin{matrix} \text{Women} & \text{Men} & \text{Children} \end{matrix}$$
$$T = \begin{bmatrix} 12\,745 & 11\,380 & 4602 \end{bmatrix}$$

(f) Find a 1×3 matrix D that shows the discounted selling price of each type of watch.

At the Richmond store, Donna, the sales manager, puts the watches into a safe.

To remember the combination for the safe, Donna uses the following clues:

- The 1st number equals the sum of the other two numbers.
- The 2nd number added to twice the 3rd number is 24 more than the 1st number.
- When the 1st and 3rd numbers are added, the result is twice the 2nd number.

(g) Use matrix methods to find the combination for the safe.

(a) Richmond: $\begin{bmatrix} 90 & 40 & 50 \\ 100 & 80 & 23 \\ 50 & 60 & 10 \end{bmatrix}$;

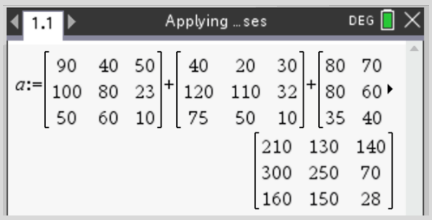
South Yarra: $\begin{bmatrix} 40 & 20 & 30 \\ 120 & 110 & 32 \\ 75 & 50 & 10 \end{bmatrix}$; Geelong: $\begin{bmatrix} 80 & 70 & 60 \\ 80 & 60 & 15 \\ 35 & 40 & 8 \end{bmatrix}$.

(b) (i) Matrix A is the sum of the above 3 matrices.

On a **Calculator** page, assign A as follows:

- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [book] [5] and select the **m-by-n Matrix** template.
- Set number of rows to 3 and number of columns to 3.
- Determine A as shown.

$$A = \begin{bmatrix} 210 & 130 & 140 \\ 300 & 250 & 70 \\ 160 & 150 & 28 \end{bmatrix}$$



Note: *The **Define** command ([menu] > **Actions** > **Define**) or the **Store** command ([ctrl] [var] to access [sto→]) can be used.*

(ii) Element a_{23} is element $(2,3)$ in A. To access element $(2,3)$, enter as shown. ($a_{23} = 70$)

There are 70 *Standard* children watches allocated across the three stores.

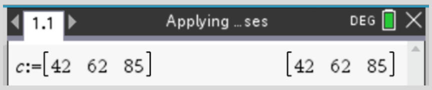
Note: *The first element in* $[2,3]$ *indicates the row number and the second element indicates the column number.*

... continued

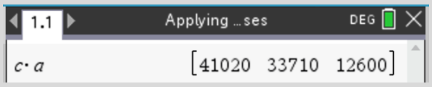
Applying matrix equations and inverses (continued)

(c) $C = \begin{bmatrix} 42 & 62 & 85 \end{bmatrix}$

- Press ctrl ⊞ to access the **Assign** := command.
- Press 📖 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 1 and the number of columns to be 3.
- Enter as shown.



$$CA = \begin{bmatrix} 42 & 62 & 85 \end{bmatrix} \begin{bmatrix} 210 & 130 & 140 \\ 300 & 250 & 70 \\ 160 & 150 & 28 \end{bmatrix}$$
$$= \begin{bmatrix} 41020 & 33\,710 & 12\,600 \end{bmatrix}$$



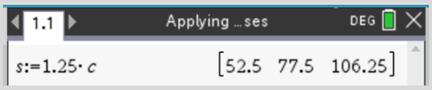
The elements in matrix CA correspond to the total wholesale price ($) paid for women's, men's and children's watches.

(d) To make a 25% profit, each watch must be sold at 1.25 times its wholesale price.

Hence $S = 1.25C$.

On a **Calculator** page, assign S as follows:

- Press ctrl ⊞ to access the **Assign** := command.
- Enter as shown.

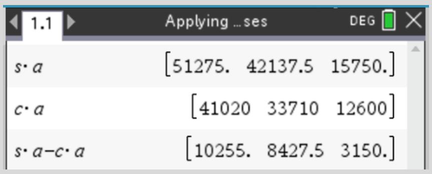


$$S = 1.25\begin{bmatrix} 42 & 62 & 85 \end{bmatrix}$$
$$= \begin{bmatrix} 52.5 & 77.5 & 106.25 \end{bmatrix}$$

(e) The total profit, P, is the total revenue from the watches when sold minus the total wholesale price paid.

Hence $P = SA - CA$.

$$SA = \begin{bmatrix} 52.5 & 77.5 & 106.25 \end{bmatrix} \begin{bmatrix} 210 & 130 & 140 \\ 300 & 250 & 70 \\ 160 & 150 & 28 \end{bmatrix}$$
$$= \begin{bmatrix} 51275 & 42137.5 & 15\,750 \end{bmatrix}$$



$$P = SA - CA$$
$$= \begin{bmatrix} 51275 & 42137.5 & 15\,750 \end{bmatrix} - \begin{bmatrix} 41020 & 33\,710 & 12\,600 \end{bmatrix}$$
$$= \begin{bmatrix} 10\,255 & 8427.5 & 3150 \end{bmatrix}$$

Alternatively:

$$P = SA - CA$$
$$= (S - C)A$$

$$S - C = \begin{bmatrix} 52.5 & 77.5 & 106.25 \end{bmatrix} - \begin{bmatrix} 42 & 62 & 85 \end{bmatrix}$$
$$= \begin{bmatrix} 10.5 & 15.5 & 21.25 \end{bmatrix}$$

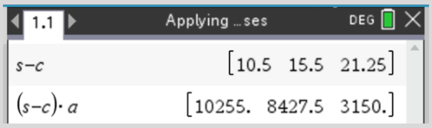
... continued

Applying matrix equations and inverses (continued)

$$P = (S - C)A$$

$$= \begin{bmatrix} 10.5 & 15.5 & 21.25 \end{bmatrix} \begin{bmatrix} 210 & 130 & 140 \\ 300 & 250 & 70 \\ 160 & 150 & 28 \end{bmatrix}$$

$$= \begin{bmatrix} 10\,255 & 8427.5 & 3150 \end{bmatrix}$$

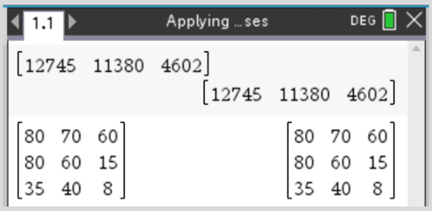


(f) D is a 1×3 matrix which represents the discounted selling prices of the watches.

Let matrix G represent the amount of stock allocated yearly to the Geelong store.

$$G = \begin{bmatrix} 80 & 70 & 60 \\ 80 & 60 & 15 \\ 35 & 40 & 8 \end{bmatrix}$$

$DG = T$ and so $D = TG^{-1}$ where $T = \begin{bmatrix} 12745 & 11380 & 4602 \end{bmatrix}$.

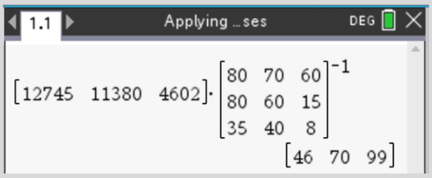


On a **Calculator** page, create G and T as follows:

- Press 5 and select the **m-by-n Matrix** template.
- For G, set the number of rows to be 3 and the number of columns to be 3.
- For T, set the number of rows to be 1 and the number of columns to be 3.
- Enter as shown.

$$D = \begin{bmatrix} 12\,745 & 11\,380 & 4602 \end{bmatrix} \begin{bmatrix} 80 & 70 & 60 \\ 80 & 60 & 15 \\ 35 & 40 & 8 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 46 & 70 & 99 \end{bmatrix}$$



Matrix D shows the discounted selling price ($) of each type of watch.

Basic: $46, *Standard*: $70 and *Special*: $99.

(g) Let x be the first number, y the second number and z the third number.

The equations are:

$$x = y + z$$
$$y + 2z = x + 24$$
$$x + z = 2y$$

... continued

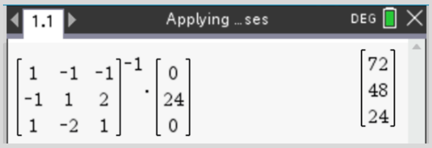
Applying matrix equations and inverses (continued)

Write these equations in standard form so they can be solved using a matrix method.

$$\begin{aligned} x - y - z &= 0 \\ -x + y + 2z &= 24 \\ x - 2y + z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 24 \\ 0 \end{bmatrix}$$

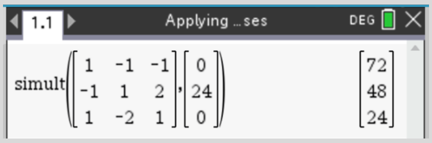
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 2 \\ 1 & -2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 24 \\ 0 \end{bmatrix}$$



The combination is 1st number 72, 2nd number 48 and 3rd number 24.

Alternatively, this system of linear equations can be solved using the **Simultaneous** command:

Press [menu] > **Matrix & Vector** > **Simultaneous** and enter as shown.



Using permutation matrices

A permutation matrix of order *n* is an $n \times n$ (square) binary matrix in which there is exactly one '1' in each row and column.

An identity matrix is an example of a permutation matrix.

A permutation matrix is either the identity matrix or the identity matrix with one or more rows (columns) swapped around.

Examples include: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Permutation matrices can be used to rearrange the elements in another matrix.

Consider the matrix product $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$.

The position of the 1's in the permutation matrix determine the rearranging of the elements in the column matrix.

There are six possible arrangements of the letters *x*, *y* and *z*.

These are: *xyz*, *xzy*, *yxz*, *yzx*, *zxy* and *zyx*.

... continued

Using permutation matrices (continued)

Corresponding to these six arrangements are six 3×3 permutation matrices, namely:

$$P_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ P_2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\ P_3=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\ P_4=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\ P_5=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\text{ and}$$

$$P_6=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}.$$

Each of these six permutation matrices can be applied to the column matrix $\begin{bmatrix}x\\y\\z\end{bmatrix}$ to produce a rearrangement of the letters *xyz*.

Example 1

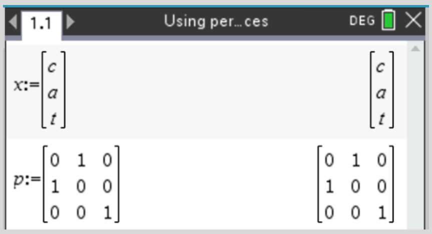
Consider the matrix $X=\begin{bmatrix}c\\a\\t\end{bmatrix}$ and the permutation matrix $P=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$.

(a) Find the matrix product *PX*.

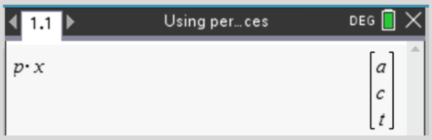
(b) Find the matrix product P^2X and interpret the result. What can be deduced about P^2?

On a **Calculator** page, assign *X* and *P* as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5] and select the **m-by-n Matrix** template.
- For *X*, set the number of rows to be 3 and the number of columns to be 1.
- For *P*, set the number of rows to be 3 and the number of columns to be 3.
- Enter as shown.



(a) Calculate *PX* as shown.



$$PX=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}c\\a\\t\end{bmatrix}$$

$$=\begin{bmatrix}a\\c\\t\end{bmatrix}$$

To find the 1st element in *PX*, note that the 1 in *P* is in the 2nd column of the 1st row.

Hence $0\times c+1\times a+0\times t=a$.

... continued

Using permutation matrices (continued)

Example 1 (continued)

To find the 2nd element in *PX*, note that the 1 in *P* is in the 1st column of the 2nd row.

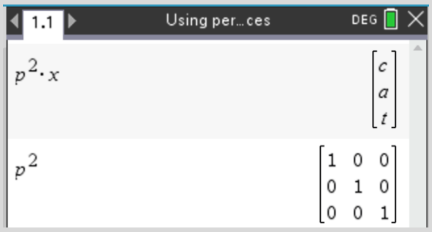
Hence $1\times c+0\times a+0\times t=c$.

To find the 3rd element in *PX*, note that the 1 in *P* is in the 3rd column of the 3rd row.

Hence $0\times c+0\times a+1\times t=t$.

(b) Calculate P^2X as shown.

$$P^2X=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}^2\begin{bmatrix}c\\a\\t\end{bmatrix}=\begin{bmatrix}c\\a\\t\end{bmatrix}$$



To leave matrix X unchanged, $P^2=I$ i.e. P^2 must be an identity matrix.

Example 2

Find a permutation matrix, P, that rearranges:

the column matrix $\begin{bmatrix}c\\o\\d\\e\end{bmatrix}$ to the column matrix $\begin{bmatrix}d\\e\\c\\o\end{bmatrix}$.

To move *d* from the 3rd row of the original matrix to the 1st row of the new column matrix, position the 1 in the 3rd column of the 1st row of *P*.

To move *e* from the 4th row of the original matrix to the 2nd row of the new column matrix, position the 1 in the 4th column of the 2nd row of *P*.

To move *c* from the 1st row of the original matrix to the 3rd row of the new column matrix, position the 1 in the 1st column of the 3rd row of *P*.

To move *o* from the 2nd row of the original matrix to the 4th row of the new column matrix, position the 1 in the 2nd column of the 4th row of *P*.

Hence

$$P=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}.$$

... continued

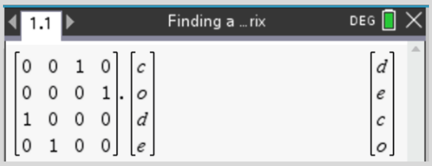
Using permutation matrices (continued)

Example 2 (continued)

To verify the result on a **Calculator** page, create P and $\begin{bmatrix} c \\ o \\ d \\ e \end{bmatrix}$.

as follows:

- Press [book] [5] and select the **m-by-n Matrix** template.
- For P, set the number of rows to be 4 and the number of columns to be 4.
- For $\begin{bmatrix} c \\ o \\ d \\ e \end{bmatrix}$, set the number of rows to be 4 and the number of columns to be 1.
- Enter as shown.



$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} c \\ o \\ d \\ e \end{bmatrix} = \begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix}$$

Using inverses of permutation matrices

If P is a permutation matrix then P^{-1} exists and $P^{-1} = P^T$ where P^T is the transpose of P.

A 4×4 permutation matrix $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ applied to a 4×1 matrix X gives $\begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix}$, a column matrix. Determine matrix X.

$PX = \begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix}$ where $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

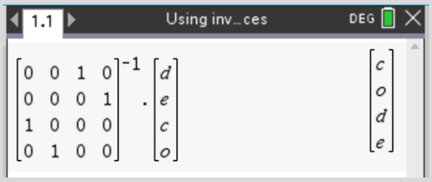
Hence either $X = P^{-1}\begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix}$ or $X = P^T\begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix}$ (where $P^{-1} = P^T$).

Using inverses of permutation matrices (continued)

On a **Calculator** page, create P and $\begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix}$ as follows:

- Press 📖 5 and select the **m-by-n Matrix** template.
- For P, set the number of rows to be 4 and the number of columns to be 4.
- For $\begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix}$, set the number of rows to be 4 and the number of columns to be 1.
- Enter as shown.

$$X = P^{-1}\begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1}\begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix} = \begin{bmatrix} c \\ o \\ d \\ e \end{bmatrix}$$

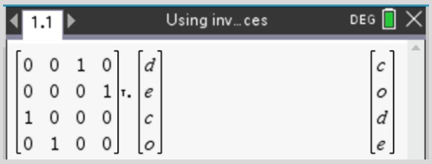


Alternatively, $X = P^{T}\begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix}$.

To create the transpose of P:

- Press menu > **Matrix & Vector** > **Transpose**.
- Enter as shown.

$$X = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{T}\begin{bmatrix} d \\ e \\ c \\ o \end{bmatrix} = \begin{bmatrix} c \\ o \\ d \\ e \end{bmatrix}$$



***Note:** The superscript **T** can also be accessed from the symbols palette (ctrl 📖) or 📖 T.*

Using dominance matrices

A dominance matrix is an $n \times n$ (square) binary matrix in which the 1s represent one-step dominances between the members of a group, for example, between players or teams.

A round-robin tournament is one in which each player or team plays every other player or team once.

Dominance matrices are used to rank players or teams in round robin tournaments.

A round robin netball tournament consists of five teams: P, Q, R, S and T.

Each team plays the other four teams once and the results of the tournament are as follows:

- P defeated R and S
- Q defeated P, R and T
- R defeated S
- S defeated Q
- T defeated P, R and S

(a) Construct a one-step dominance matrix, D.

(b) Construct a two-step dominance matrix, D^2.

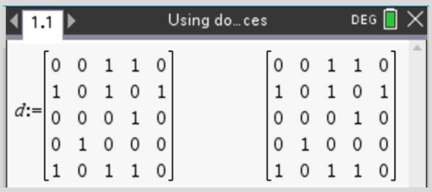
(c) Use the dominance scores from the matrix $D + D^2$ to rank the five teams.

(a) Construct the one-step dominance matrix, D.

$$D = \begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{One-step} \\ 2 \\ 3 \\ 1 \\ 1 \\ 3 \end{array}$$

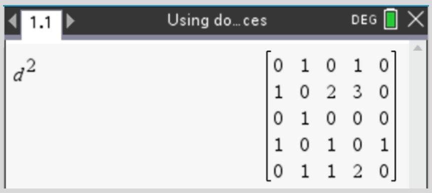
On a **Calculator** page, assign D as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 5 and the number of columns to be 5.
- Enter as shown.



(b) Calculate the two-step dominance matrix, D^2 as shown.

$$D^2 = \begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{Two-step} \\ 2 \\ 6 \\ 1 \\ 3 \\ 4 \end{array}$$

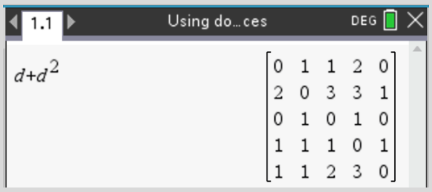


... continued

Using dominance matrices (continued)

(c) Calculate the matrix $D+D^2$ as shown.

$$D+D^2=\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array}\begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{bmatrix} \end{array}\begin{array}{c} \text{Total} \\ 4 \\ 9 \\ 2 \\ 4 \\ 7 \end{array}$$



The team with the highest total dominance score in matrix $D+D^2$ has the highest ranking.

By this measure, Q (9) is the highest ranking team, followed by T (7), P and S (tied on 4) and R (2).

***Note:** This measure of dominance treats two-step dominances to be as important as one-step dominances. This can create odd results.*

4.2. Transition matrices

Transition matrices are used to predict what will happen in the future based on what has happened previously and the likelihood of an event occurring given what has happened previously.

A transition matrix T is useful when we wish to model situations where:

- the conditions or states are clearly defined sets.
- there is a transition from one state to the next.
- the next state depends only on the previous one.

The initial state matrix S_0 gives the numbers or proportions of the objects initially in each state.

The state matrix after n transitions can be found using $S_n = T^n S_0$.

If the previous state is known, the next state can be found using $S_{n+1} = TS_n$.

As the number of transitions increases indefinitely, the proportions in each category will approach a steady state.

In other words, as n gets large, S_n stabilises to a steady state.

Applying transition matrices

A survey of coffee buyers who buy three brands of coffee is conducted.

The following table of proportions shows the results of the survey.

	Current purchase		
Next purchase	Brand A	Brand B	Brand C
Brand A	0.8	0.2	0.15
Brand B	0.1	0.7	0.2
Brand C	0.1	0.1	0.65

(a) Find the proportion of coffee buyers whose current (i.e. first) purchase is brand B and who are expected to buy brand A on

(i) their next (i.e. second) purchase.

(ii) each of their next two purchases (i.e. their second and third purchases).

(iii) their third purchase.

A shop has 180 regular customers who buy these three brands of coffee.

Each customer buys one jar of coffee each week.

Initially (in week one), 60 customers buy a jar of brand A, 60 customers buy a jar of brand B and the remaining customers buy a jar of brand C.

(b) If a customers' subsequent buying patterns follows the proportions in the table above, find the number of jars of coffee of each brand that the shopkeeper can expect to sell in

(i) week two.

(ii) week three.

(c) In the long term, i.e. after many weeks, how many jars of each brand does the shopkeeper expect to sell?

... continued

Applying transition matrices (continued)

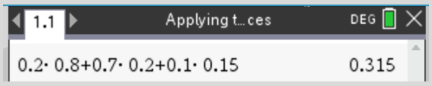
(a) (i) Reading directly from the table, the proportion of coffee buyers whose current (i.e. first) purchase is brand B and who are expected to buy brand A on their next (i.e. second) purchase is 0.2.

(ii) The proportion of coffee buyers whose current (i.e. first) purchase is brand B and who are expected to buy brand A on each of their next two purchases (i.e. their second and third purchases) is $0.2 \times 0.8 = 0.16$.

(iii) Using a tree diagram for example, the possible outcomes for coffee buyers whose current (i.e. first) purchase is brand B and who are expected to buy brand A on their third purchase are BAA, BBA or BCA.

The proportion of coffee buyers whose current (i.e. first) purchase is brand B and who are expected to buy brand A on their third purchase is

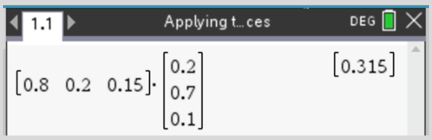
$(0.2 \times 0.8) + (0.7 \times 0.2) + (0.1 \times 0.15) = 0.315$.



Alternatively, using the 1×3 row matrix $\begin{bmatrix} 0.8 & 0.2 & 0.15 \end{bmatrix}$ for brand A (3rd purchase) and the 3×1 column matrix $\begin{bmatrix} 0.2 \\ 0.7 \\ 0.1 \end{bmatrix}$ for brand B (1st purchase):

- Press 🕮 5 and select the **m-by-n Matrix** template.
- For the row matrix, set the number of rows to be 1 and the number of columns to be 3.
- For the column matrix, set the number of rows to be 3 and the number of columns to be 1.
- Enter as shown.

$$\begin{bmatrix} 0.8 & 0.2 & 0.15 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.7 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.315 \end{bmatrix}$$



(b) (i) The transition matrix is $T = \begin{bmatrix} 0.8 & 0.2 & 0.15 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.65 \end{bmatrix}$.

The initial state matrix is $S_0 = \begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix}$.

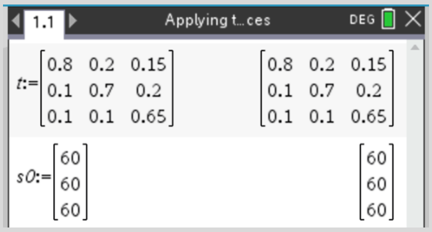
On a **Calculator** page, assign T and S_0 as follows:

- Press ctrl ⊞ to access the **Assign** [:=] command.
- For T, press 🕮 5 and select the **m-by-n Matrix** template.

... continued

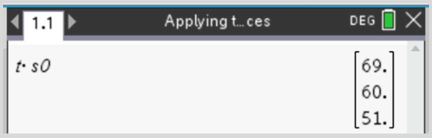
Applying transition matrices (continued)

- Select the number of rows to be 3 and the number of columns to be 3.
- For S_0, press [catalog] [5] and select the **m-by-n Matrix** template.
- Select the number of rows to be 3 and the number of columns to be 1.
- Enter as shown.



The number of jars of coffee of each brand that the shopkeeper can expect to sell in week two is given by:

$$S_1 = TS_0 = \begin{bmatrix} 0.8 & 0.2 & 0.15 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.65 \end{bmatrix}\begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix} = \begin{bmatrix} 69 \\ 60 \\ 51 \end{bmatrix}$$

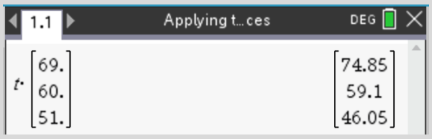


The expected sales of each brand in week two are:

brand A: 69; brand B: 60; brand C: 51.

(ii) The number of jars of coffee of each brand that the shopkeeper can expect to sell in week three is given by:

$$S_2 = TS_1 = \begin{bmatrix} 0.8 & 0.2 & 0.15 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.65 \end{bmatrix}\begin{bmatrix} 69 \\ 60 \\ 51 \end{bmatrix} = \begin{bmatrix} 74.85 \\ 59.1 \\ 46.05 \end{bmatrix}$$



Alternatively, $S_2 = T^2 S_0$.

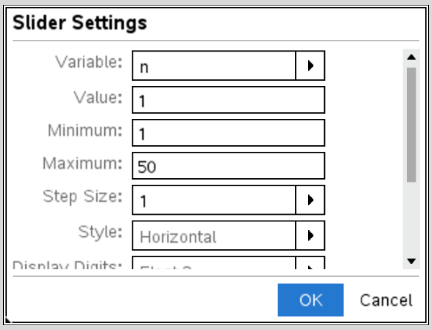
The expected sales of each brand in week three are:

brand A: 75; brand B: 59; brand C: 46.

(c) To investigate the number of jars of each brand the shop expects to sell in the long term, we can use an alternative approach based on the powers of the transition matrix.

On a **Notes** page:

Insert a **Slider** to control the value of n as follows:



- Press [menu] > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.

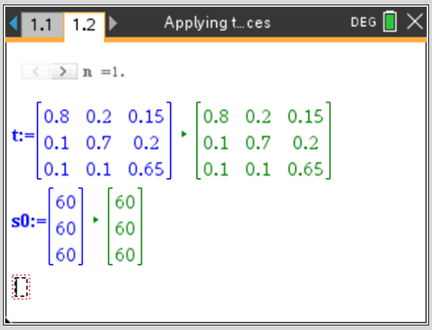
Insert a **Maths Box** to assign T and S_0 as follows:

- Press [menu] > **Insert** > **Maths Box**.

*Note: Alternatively, to insert a **Maths Box**, press* [ctrl] [M].

Assign T as follows:

- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [catalog] [5] and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 3.



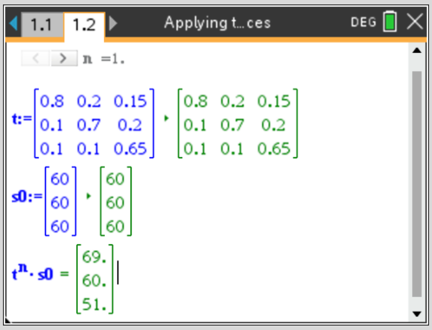
... continued

- Enter as shown.

Applying transition matrices (continued)

Assign S_0 as follows:

- Press ctrl to access the **Assign** := command.
- Press 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 1.
- Enter as shown.



Assign S_n as $T^n S_0$:

- Press ctrl M to insert a **Maths Box**
- Press ctrl to access the **Assign** := command and enter as shown.

To change the display of a **Maths Box**, for example, to display an equals sign:

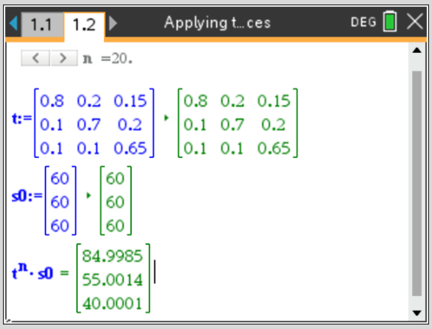
- Click on the **Maths Box**.
- Press menu > **Maths Box Options** > **Maths Box Attributes**.
- Press tab to highlight the **Insert Symbol** field.
- Press ▶ and select =.

Notes:
*1. **Maths Box Attributes** can also be accessed within a **Maths Box** by pressing ctrl menu.*
*2. If desired, change the font size on a Notes page by pressing menu > **Format** > **Format text**.*

(c) Click on the slider to change the value of n and observe the behaviour of successive state matrices. Note that you can also manually change the value of n in the slider by clicking on the slider value and editing it.

For example, consider $n = 20$.

$$T^{20}S_0 = \begin{bmatrix} 84.9985 \\ 55.0014 \\ 40.0001 \end{bmatrix}$$



In the long term, the number of jars of each brand the shopkeeper expects to sell appear to be:

brand A: 85; brand B: 55; brand C: 40.

Note: *While no change in successive state matrices is strongly indicative that steady state has been reached, we should always check by finding a state matrix far into the future.*

Solving problems involving Leslie matrices

Leslie matrices are transition matrices that are used to model changes in the sizes of different age categories within a population over time.

For example, they can be used to model and analyse an insect population comprising eggs, juveniles and adults.

A Leslie matrix takes into account two factors for the females in each age category: the birth (fecundity) rate, b_i and the survival rate, s_i, where i is the number of the age category.

Leslie matrices can be represented by transition (life cycle) diagrams that show the transitions between states.

The population state matrix, S_n, is a column matrix that lists the number in each age category at a given time.

The initial population state matrix is denoted by S_0.

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix} \text{ is an } m \times m \text{ Leslie matrix where:}$$

- m is the number of age categories being considered.
- s_i is the survival rate.
- b_i is the birth (fecundity) rate.

The state matrix, S_n, is an $m \times 1$ matrix representing the number of each age category after n time periods. This is calculated using the recursive formula

- $S_{n+1} = LS_n$ where S_0 is the initial state (population) matrix,

or with the explicit formula

- $S_n = L^n S_0$ where S_0 is the initial state (population) matrix.

Consider the following Leslie matrix L and initial population matrix S_0:

$$L = \begin{bmatrix} 0 & 2.5 & 0.5 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} 1000 \\ 500 \\ 125 \end{bmatrix}$$

(a) Determine S_1, S_5, S_{14} and S_{15}.

(b) Find the total population after 15 time periods.

(c) Find the ratio of the total population for the 15th and 16th time periods.

(d) Plot the ratio of the total population for successive time periods.

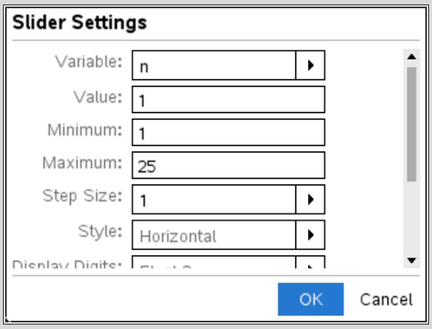
... continued

Solving problems involving Leslie matrices (continued)

On a **Notes** page:

Insert a **Slider** to control the value of n as follows:

- Press menu > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.



Insert a **Maths Box** as follows:

- Press menu > **Insert** > **Maths Box**.

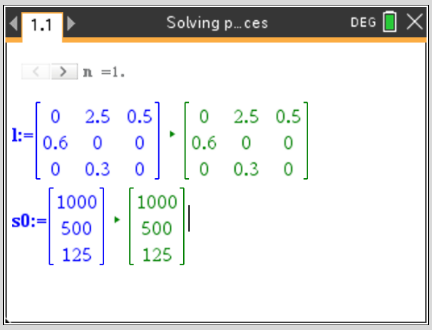
Note: *Alternatively, to insert a* ***Maths Box****, press* ctrl M.

Assign L as follows:

- Press ctrl to access the **Assign** := command.
- Press 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 3.
- Enter as shown.

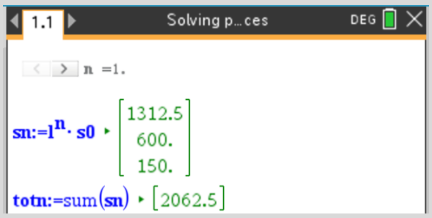
Assign S_0 as follows:

- Press ctrl to access the **Assign** := command.
- Press 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 3 and the number of columns to be 1.
- Enter as shown.



Assign S_n as $L^n \cdot S_0$:

- Press ctrl M to insert a **Maths Box**
- Press ctrl to access the **Assign** := command and enter as shown.



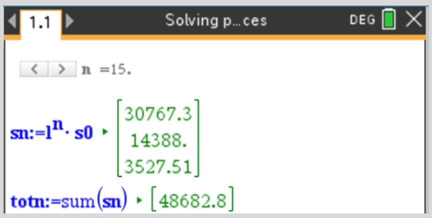
To add the three age category populations, enter the formula $totn := \text{sum}(sn)$ as shown.

Note: *The* ***Sum*** *command can be typed in using the keypad, or by pressing* S *and scrolling to 'sum'.*

(a) Click on the slider to change the value of n and hence determine S_1, S_5, S_{14} and S_{15}.

$$S_1 = \begin{bmatrix} 1312.5 \\ 600 \\ 150 \end{bmatrix},\ S_5 = \begin{bmatrix} 3229.88 \\ 1488.38 \\ 370.575 \end{bmatrix},\ S_{14} = \begin{bmatrix} 23\,979.9 \\ 11758.4 \\ 2742.76 \end{bmatrix},$$

$$S_{15} = \begin{bmatrix} 30\,767.3 \\ 14\,388.0 \\ 3527.51 \end{bmatrix}$$



... continued

Solving problems involving Leslie matrices (continued)

(b) The total population after 15 time periods is found by adding the three age category populations in S_{15}.

The total population is approximately 48 683.

(c) To calculate the state matrix after $n+1$ time periods:

- Enter the formula $\boldsymbol{snplus1 := l^{n+1} \cdot s0}$.

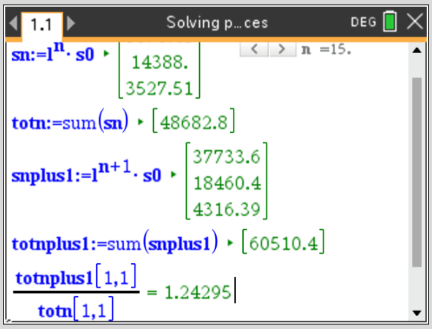
To add the three age populations after $n+1$ time periods:

- Enter the formula $\boldsymbol{totnplus1 := \mathrm{sum}(snplus1)}$.

To find the successor ratio for the total population between consecutive time periods:

- Enter the formula $\dfrac{\boldsymbol{totnplus1[1,1]}}{\boldsymbol{totn[1,1]}}$.

Use the slider to set $n = 15$. The percentage increase between the 15th and 16th time periods is approximately 24.3%.



***Note:** To change the display of a **Maths Box**, for example, to display an equals sign:*

- *Click on the **Maths Box**.*
- *Press [menu] > **Maths Box Options** > **Maths Box Attributes**.*
- *Press [tab] to highlight the **Insert Symbol** field.*
- *Press ▸ and select =.*

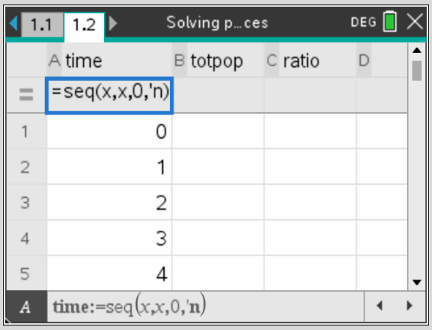
***Maths Box Attributes** can also be accessed within a **Maths Box** by pressing [ctrl] [menu].*

(d) To calculate the values for the successor ratio, add a **Lists & Spreadsheet page**, then:

- In the column A heading cell enter the variable ***time***.
- In the column B heading cell enter the variable ***totpop***.
- In the column C heading cell enter the variable ***ratio***.

To generate the values for the time period in the column A formula cell:

- Press [=] [📖] [S], scroll down and select **seq(**.
- Enter $\boldsymbol{time := \mathrm{seq}(x, x, 0, 'n)}$.



***Notes**:*

***1.** The syntax for expressing a sequence as a list is **seq(Expression, Variable, Low, High[,Step])**. The default value for **Step** is 1.*

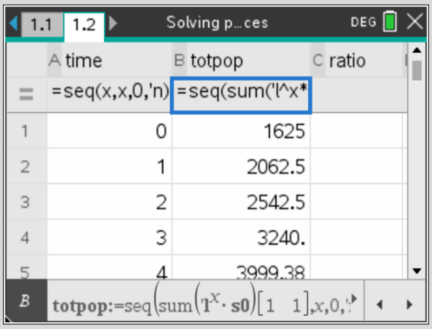
***2.** The symbol **'** in **'n** specifies n as a variable reference (press [?!▸] to access the **'** symbol). Otherwise, TI-Nspire CX II CAS will consider **n** as a column reference. If the **'** symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column. An alternative is to use the [var] key if the variable is already assigned/defined.*

... continued

Solving problems involving Leslie matrices (continued)

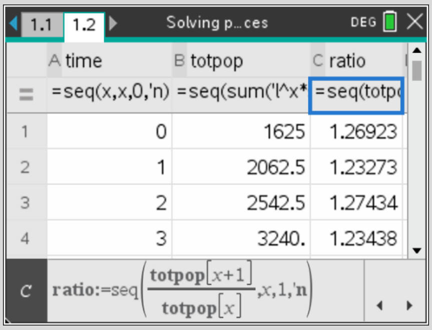
To generate the total population after n time periods, enter the following in the column B formula cell:

- Press [=] [📖] [S], scroll down and select **seq(**.
- Enter $\boldsymbol{totpop := \mathbf{seq}\left(\mathbf{sum}('l^x \cdot s0)[1,1], x, 0, 'n\right)}$.



To generate the ratio for the $n+1$ and n time periods, enter the following in the column C formula cell:

- Press [=] [📖] [S], scroll down and select **seq(**.
- Enter $\boldsymbol{ratio := \mathbf{seq}\left(\frac{totpop[x+1]}{totpop[x]}, x, 1, 'n\right)}$.

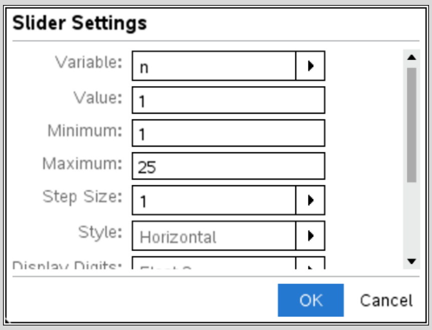


To construct a time series plot of ***ratio***, add a **Data & Statistics** page, then:

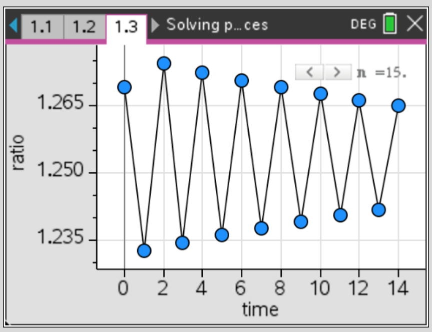
- Press [tab] and select ***time*** on the horizontal axis.
- Press [tab] and select ***ratio*** on the vertical axis..
- Press [menu] **> Plot Properties > Connect Data Points**.

To add a slider that allows the user to alter the number of values of the successor ratio to be plotted on the **Data & Statistics** page:

- Press [menu] **> Actions** > **Insert Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.



From the time series plot, it appears that the value of the ratio is oscillating and converging to a value of just over 1.25 (approaching an increase of just over 25% in the total population for each time period).



4.3. Networks and decision mathematics

Matrices are used to represent graphs, digraphs, networks and their applications.

See Section 1.4.2 (page 43) for an example of using matrices to determine one-step and two-step pathways in a network.

Using adjacency matrices to represent an undirected graph

An adjacency matrix, A, is an $n \times n$ matrix that shows the number of connections between the vertices of a graph.

A loop is a single edge connecting a vertex to itself.

Loops are counted as one edge.

In a graph, a walk is a sequence of edges that connect successive vertices.

A walk starts at one vertex and follows any route to finish at another vertex.

Walks are specified by listing the vertices in the order they are visited.

The adjacency matrix, A, of a graph G, with vertices P, Q, R, S and T is given by

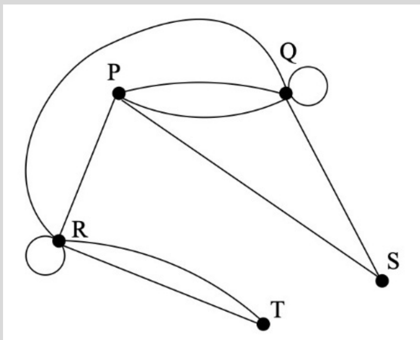
$$A = \begin{array}{c} \\ \text{P} \\ \text{Q} \\ \text{R} \\ \text{S} \\ \text{T} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{P} & \text{Q} & \text{R} & \text{S} & \text{T} \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} \end{array}.$$

(a) Draw the graph G.

(b) Find the number of walks of length 5 from P to Q.

(c) Determine the pairs of distinct vertices that have more than 15 walks of length 3 between them.

(a) Graph G is shown right:



From A, for example, there are two edges connecting P and Q, one edge connecting P and R and one edge connecting P and S.

The main diagonal of A indicates there is a loop (single edge) connecting Q to itself and R to itself.

(b) To find the number of walks of length 5 from P to Q, we need to calculate A^5 where:

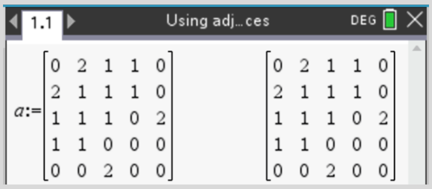
$$A = \begin{array}{c} \\ \text{P} \\ \text{Q} \\ \text{R} \\ \text{S} \\ \text{T} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{P} & \text{Q} & \text{R} & \text{S} & \text{T} \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

... continued

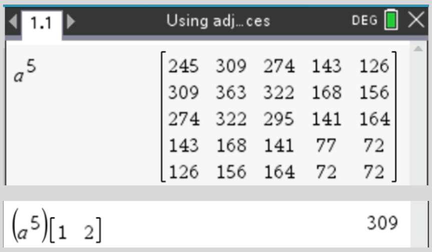
Using adjacency matrices to represent an undirected graph (continued)

On a **Calculator** page, assign A as follows:

- Press ctrl to access the **Assign** := command.
- Press 5 and select the **m-by-n Matrix** template.
- Set the number of rows to be 5 and the number of columns to be 5.
- Enter as shown.



$$A^5 = \begin{bmatrix} 245 & 309 & 274 & 143 & 126 \\ 309 & 363 & 322 & 168 & 156 \\ 274 & 322 & 295 & 141 & 164 \\ 143 & 168 & 141 & 77 & 72 \\ 126 & 156 & 164 & 72 & 72 \end{bmatrix}$$



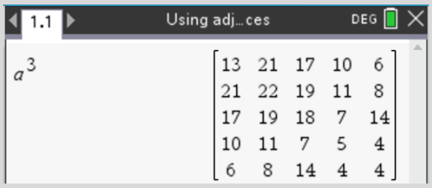
The number of walks of length 5 from P to Q is given by the element in row 1 and column 2 of A^5.

To access this element, enter as shown.

Hence there are 309 walks of length 5 from P to Q.

Note: *The first element in* $[1 \quad 2]$ *indicates the row number and the second element indicates the column number.*

(c) To determine the pairs of distinct vertices that have more than 15 walks of length 3 between them, we need to calculate A^3.

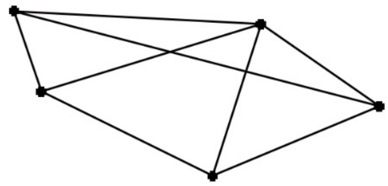


$$A^3 = \begin{bmatrix} 13 & 21 & 17 & 10 & 6 \\ 21 & 22 & 19 & 11 & 8 \\ 17 & 19 & 18 & 7 & 14 \\ 10 & 11 & 7 & 5 & 4 \\ 6 & 8 & 14 & 4 & 4 \end{bmatrix}$$

To identify these pairs of distinct vertices, we look for elements in A^3 that are >15 and not on the leading (main) diagonal.

The pairs of distinct vertices are PQ (21), PR (17) and QR (19).

Demonstrating planarity with the Geometry App

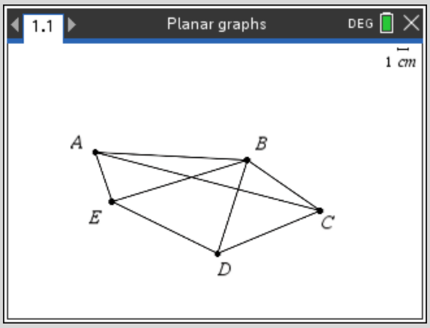


The Geometry App is a handy teacher tool for constructing network graphs. They are useful for creating graphs in which the vertices can be moved to help visualise whether a graph is planar or not.

Consider the undirected graphs shown right – construct this appropriately labelled graph using the **Geometry** App, and then drag a vertex as required to check whether this graph is planar.

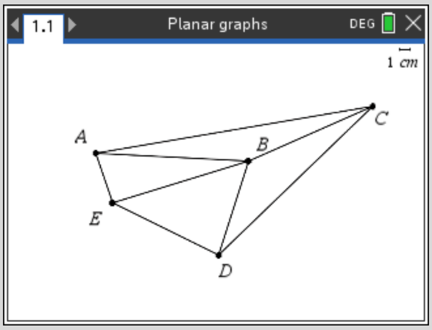
To construct the undirected graph, on a **Geometry** page:

- Press menu > **Points & Lines** > **Point**.
- Click on the **Geometry** page in five locations to represent the vertices in the diagram above.
- Press esc to exit the **Point** tool.
- Press menu > **Points & Lines** > **Segment**.
- Click on each pair of vertices that are joined by an edge (as per the above diagram)
- Press esc to exit the **Segment** tool.
- Press menu > **Points & Lines** > **Segment**.



To label the vertices A to E:

- Click on a vertex and then press ctrl menu and select **Label** from the pop-up menu.
- Enter the label text 'A'.
- Repeat the above for the remaining vertices as shown right.

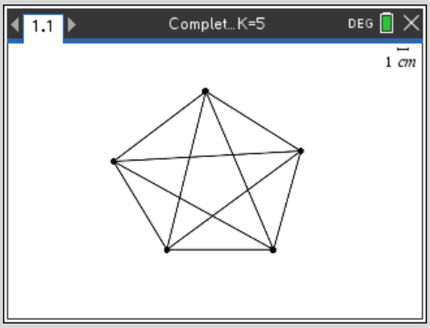


To verify that the graph is planar:

- Click and drag Point C upwards and to the right so that the line segment AC no longer appears to be 'over' the line segments BD and BE.

Answer: By moving the location of point C, we can see more clearly that there are $V = 5$ vertices, $E = 8$ edges and $R = 5$ regions, and that $V - E + R = 5 - 8 + 5 = 2$ (and so the graph is planar).

***Note**: It is also useful to use such constructions to highlight that a graph might not be planar. For example, the complete graph of order five or greater is not planar. The complete graph of order five is shown right.*



Using a program to execute the Hungarian algorithm

Note*: There are several useful and publicly available TI-Nspire CX II CAS files that can be used as tools for completing mathematical algorithms. One such file is* ***Hungarian Algorithm.tns****, which contains a program which finds the optimal allocation and cost using the Hungarian algorithm. This can be found at https://www.ticalc.org/archives/files/fileinfo/470/47035.html or by performing an internet search for "Hungarian Algorithm.tns". Once downloaded to a computer, it can be transferred to a TI-Nspire CX II CAS calculator using the TI-Nspire software and cable.*

The following example problem uses this file to determine the optimal allocation and cost using the Hungarian algorithm. Note that no intermediate steps are shown.

A local government agency seeks tenders for three jobs. Three companies (Company A, B and C) submit tenders. The tender for each job in hundreds of dollars is shown in the table at right.

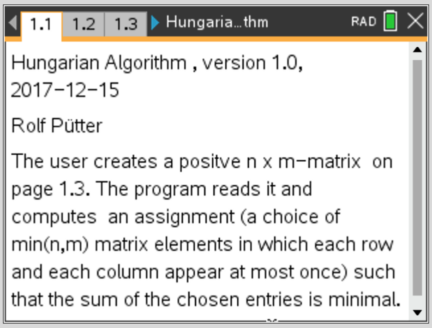
Company \ Job	1	2	3
A	15	25	12
B	18	27	11
C	16	28	14

(a) Use the Hungarian algorithm to find the optimal allocation, i.e. the allocation that minimises the total cost, with each company allocated one job.

(b) A fourth company D submits a late tender: $1700, $2500 and $1300 for jobs 1, 2, and 3 respectively. If the local government agency can look at the late tender, will this change the optimal allocation? If so, how much money will be saved, and which of the original companies will miss out?

Open the file **Hungarian Algorithm.tns**. It has three pages.

- Page 1.1 contains a brief explanation of how the underlying program works.
- Page 1.2 is where the Hungarian algorithm is completed and results will be shown.
- Page 1.3 is where users can enter an initial matrix displaying the costs for each person (rows) to complete each of the tasks (columns).

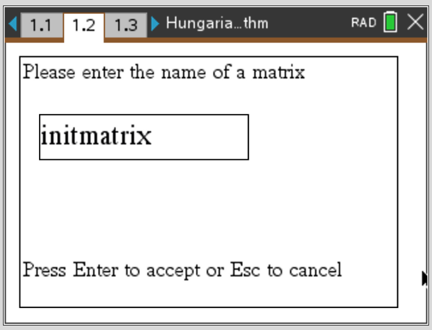


To enter an initial 3 × 3 matrix (named **initmatrix**) for which the Hungarian algorithm is to be applied:

- Press [ctrl] ▶ to navigate to Page 1.3.
- Press [template key] and select the matrix template to enter the initial 3 x 3 matrix as shown.
- Press [ctrl] [var] to store the matrix.
- Enter the text **initmatrix.**

(a) To execute the Hungarian algorithm:

- Press [ctrl] ▶ to navigate to Page 1.2.
- Press [menu] > **Matrix > User-defined** and enter the matrix name **initmatrix**.
- Press [enter] to execute all the steps of the Hungarian algorithm at once.



... continued

Using a program to execute the Hungarian algorithm (continued)

Answer: The optimal allocation can now be found, which is:

- Company A should be allocated Job 2
- Company B should be allocated Job 3
- Company C should be allocated Job 1

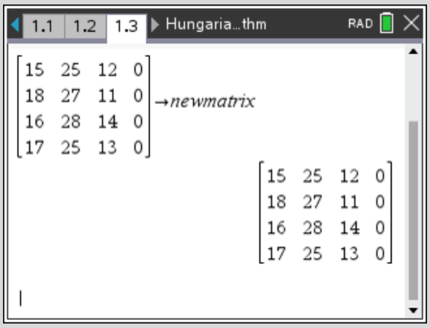
The total cost of this optimal allocation is also calculated as: 25 + 11 + 16 = 52 (i.e. $5200).

***Note**: On Page 1.2, pressing* [enter] *will automatically execute the algorithm. An alternative approach is to select cells in the matrix individually and press* [␣] *(the 'space' key). Once selected (assigned), the impact of this selection on the total allocation cost of all assignments is calculated.*

(b) If the tender from Company D is permitted, a new row of costs can be added. So that a square matrix is maintained, a dummy job (Job 4) is added.

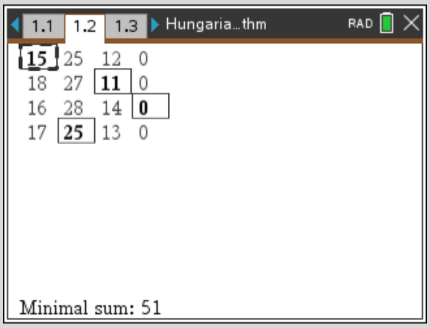
To enter a 4 × 4 matrix (named **newmatrix**) for which the Hungarian algorithm is to be applied:

- Press [ctrl] ▶ to navigate to Page 1.3.
- Press [template] and select the matrix template to enter the initial 4 x 4 matrix as shown.
- Press [ctrl] [var] to store the matrix.
- Enter the text **newmatrix.**



To execute the Hungarian algorithm for **newmatrix**:

- Press [ctrl] ◀ to navigate to Page 1.2.
- Press [menu] > **Matrix** > **User-defined** and enter the matrix name **newmatrix**.
- Press [enter] to execute all the steps of the Hungarian algorithm at once.



Answer: The optimal allocation can now be found, which is:

- Company A should be allocated Job 1
- Company B should be allocated Job 3
- Company C should be allocated Job 4 (i.e. no job)
- Company D should be allocated Job 2

The total cost of this optimal allocation is also calculated as: 15 + 11 + 25 = 51 (i.e. $5100), so $100 has been saved.

Appendix: TI-Nspire Shortcuts and Tips

(With thanks to Neale Woods. Note that a tick (✓) indicates where the shortcut is applicable. MacOS users – substitute CMD for ctrl *.)*

Shortcut	Handheld	Computer	Global	Calc	Graph	Geom	L&S	D&S	Notes	DataQ	Prog	Python	Result
ctrl A	✓	✓		✓					✓	✓	✓	✓	Select all text
ctrl B	✓	✓									✓	✓	Check Syntax & Store/Save
ctrl C	✓	✓	✓										Copy text
ctrl E	✓	✓							✓				Insert a Chem Box
ctrl F	✓	✓									✓	✓	Find
ctrl G	✓	✓			✓								Toggle Graph entry line
ctrl G	✓	✓					✓				✓	✓	Go to line/cell
ctrl H	✓	✓									✓	✓	Find and replace
ctrl I	✓	✓	✓										Insert page
ctrl J		✓	✓										Take screen capture
ctrl K	✓	✓	✓										Select page (in split screen)
ctrl L	✓	✓		✓	✓	✓			✓		✓		Display variable list
ctrl L	✓	✓					✓						Store or link to variable
ctrl M	✓	✓							✓				Insert Math Box
ctrl N	✓	✓	✓										New document
ctrl O	✓	✓	✓										Open document
ctrl P		✓	✓										Print document
ctrl R	✓	✓					✓						Recalculate
ctrl R	✓	✓									✓	✓	Checks syntax/run program
ctrl S	✓	✓	✓										Save document
ctrl T	✓	✓			✓		✓						Toggle Table/No Table
ctrl T	✓	✓									✓	✓	Comment/Uncomment
ctrl V	✓	✓	✓										Paste text
ctrl W	✓	✓	✓										Close current document
ctrl X	✓	✓	✓										Cut text
ctrl Y	✓	✓	✓										Redo
ctrl Z	✓	✓	✓										Undo

Appendix: TI-Nspire Shortcuts and Tips (continued)

(Note that a tick (✓) indicates where the shortcut is applicable. MacOS users – substitute CMD for ctrl *.)*

Shortcut	Handheld	Computer	Global	Calc	Graph	Geom	L&S	D&S	Notes	DataQ	Prog	Python	Result
ctrl 1	✓	✓		✓			✓		✓	✓	✓	✓	Jump to last line
ctrl 2	✓	✓		✓			✓		✓	✓	✓	✓	Jump to end of line/last cell
ctrl 3	✓	✓		✓			✓		✓	✓	✓	✓	Page down
ctrl 4	✓	✓	✓										Merge two pages into split screen.
ctrl 6	✓	✓	✓										Convert split screen into two pages
ctrl 7	✓	✓		✓			✓		✓	✓	✓	✓	Jump to first line
ctrl 8	✓	✓		✓			✓		✓	✓	✓	✓	Jump to start of line/first cell
ctrl 9	✓	✓		✓			✓		✓			✓	Page up
ctrl ␣	✓	✓	✓										Underscore
ctrl tab	✓		✓										Toggle b/w split screen windows
ctrl tab		✓	✓										Toggle b/w open documents
tab	✓	✓		✓	✓		✓	✓	✓	✓	✓		Move through fields or zones
tab	✓	✓			✓								Display graph entry line
tab	✓	✓										✓	Indent
⇧shift ◀▶▲▼	✓	✓	✓										Highlight selected text
⇧shift tab	✓	✓			✓		✓	✓	✓	✓			Move back through fields or zones
⇧shift tab	✓	✓										✓	Remove indent
⇧shift +	✓			✓	✓		✓	✓	✓		✓		Derivative
⇧shift −	✓			✓	✓		✓	✓	✓		✓		Integral
⇧shift ↵	✓			✓	✓			✓	✓		✓		Add a column to a matrix
⇧shift enter		✓		✓	✓			✓	✓		✓		Add a column to a matrix
↵	✓			✓	✓			✓	✓		✓		Add a row to a matrix
ALT enter		✓		✓	✓			✓	✓		✓		Add a row to a matrix
P	✓	✓			✓	✓							Add a point

Appendix: TI-Nspire Shortcuts and Tips (continued)

From the Handheld or Computer Keyboard

To enter this:	Type this shortcut:
π	**pi**
θ	**theta**
∞	**infinity**
≤	**<=**
≥	**>=**
≠	**/=**
⇒ (logical implication)	**=>**
⇔ (logical double implication, XNOR)	**<=>**
→ (store operator)	**=:**
| | (absolute value)	**abs(...)**
√()	**sqrt(...)**
Σ() (Sum template)	**sumSeq(...)**
Π() (Product template)	**prodSeq(...)**
$\sin^{-1}()$, $\cos^{-1}()$, ...	**arcsin(...), arccos(...), ...**
ΔList()	**deltaList(...)**
ΔtmpCnv()	**deltaTmpCnv(…)**

From the Computer Keyboard

To enter this:	Type this shortcut:
e (natural log base e)	**@e**
E (scientific notation)	**@E**
ᵀ (transpose)	**@t**
ʳ (radians)	**@r**
° (degrees)	**@d**
ᵍ (gradians)	**@g**
∠ (angle)	**@<**
▶ (conversion)	**@>**
▶ Decimal, ▶ approxFraction(), and so on.	**@>Decimal, @>approxFraction(), and so on.**
*c*1, *c*2, … (constants)	**@c1, @c2, …**
*n*1, *n*2, … (integer constants)	**@n1, @n2, …**
i (imaginary constant)	**@i**

Useful functions/commands available in the Catalog not available in the menus.

Function/Command name	Function/Command purpose
and	Boolean ‘and’, useful for specifying restrictions.
domain(expr,var)	Displays the domain of a function.
euler(…)	Generates a table of values using Euler’s method.
isPrime(…)	Displays ‘true’ if prime and ‘false’ if composite.
true	Displays ‘true’ if two expressions are equivalent.

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$$V_{n+1} = RV_n + d$$

$$z = \frac{x - \bar{x}}{s_x}$$

$$\sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$

Receiver

Sender

	A	B	C	D	E
A	0	1	0	0	1
B	1	0	0	0	1
C	1	0	0	0	1
D	1	0	1	0	1
E	1	0	1	1	0

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