

Monday Night Calculus

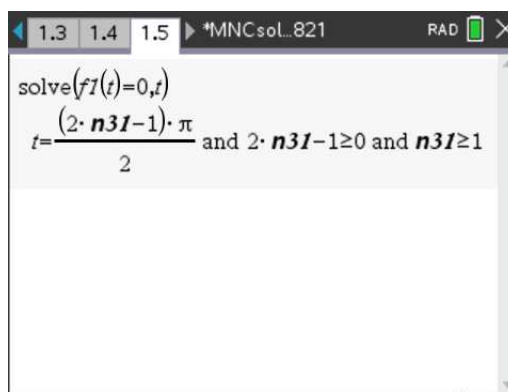
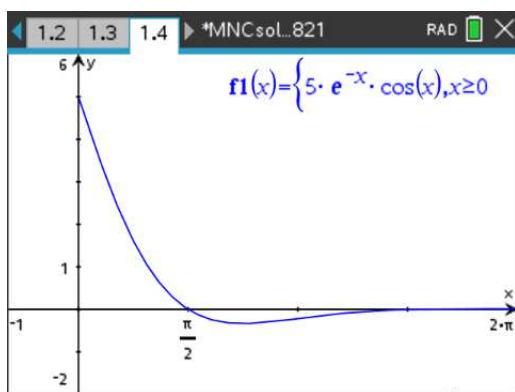
Rectilinear Motion

Exercises

1. Two objects oscillate along a vertical axis, starting at the same initial position $y = 5$ at time $t = 0$. The position of Object A at time $t, t \geq 0$, is given by $y_1(t) = 5e^{-t} \cos t$, and the position of Object B at time $t, t > 0$, is given by $y_2(t) = \frac{5 \sin t}{t}$.

- (a) Find the first time $t_a > 0$ at which Object A has position 0. What is Object A's velocity, speed, and acceleration at that time?

Consider a graph of Object A's position.



$$y_1(t) = 5e^{-t} \cos t = 0 \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}$$

$$v_1(t) = 5[-e^{-t} \cos t + e^{-t}(-\sin t)]$$

Product Rule

$$= -5e^{-t}(\cos t + \sin t)$$

Simplify

$$a_1(t) = -5[-e^{-t}(\cos t + \sin t) + e^{-t}(-\sin t + \cos t)]$$

Product Rule

$$= -5e^{-t}[-\cos t - \sin t - \sin t + \cos t]$$

Factor

$$= -5e^{-t}(-2 \sin t) = 10e^{-t} \sin t$$

Simplify

Velocity of Object A at time $t_a = \frac{\pi}{2}$:

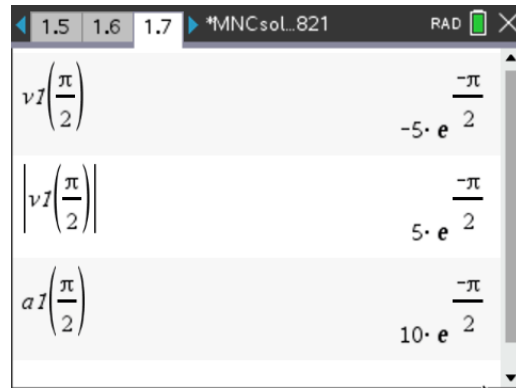
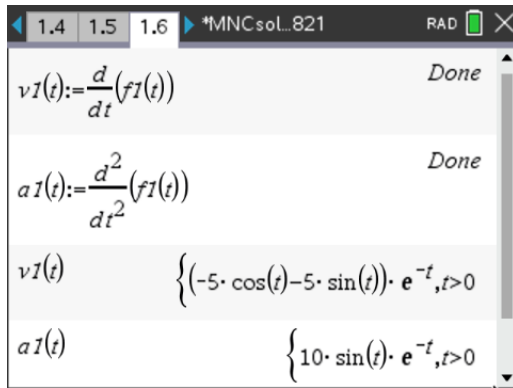
$$v_1\left(\frac{\pi}{2}\right) = -5e^{-\pi/2} \left[\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \right] = -5e^{-\pi/2}(0 + 1) = -5e^{-\pi/2}$$

Speed of Object A at time $t_a = \frac{\pi}{2}$:

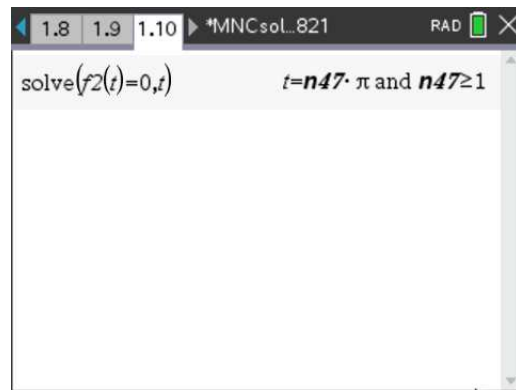
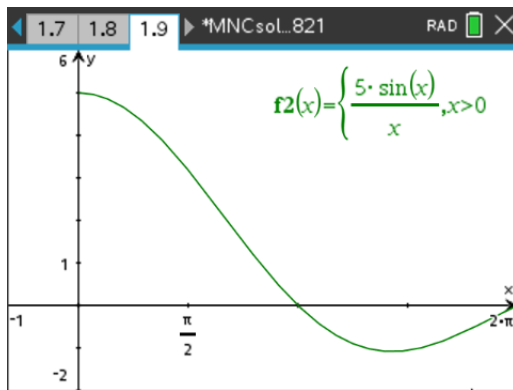
$$\left| v_1\left(\frac{\pi}{2}\right) \right| = \left| -5e^{-\pi/2} \right| = 5e^{-\pi/2}$$

Acceleration of Object A at time $t_a = \frac{\pi}{2}$:

$$a_1\left(\frac{\pi}{2}\right) = 10e^{-\pi/2}$$



- (b) Find the first time $t_b > 0$ at which Object B has position 0. What is Object B's velocity, speed, and acceleration at that time?
Consider a graph of Object B's position.



$$y_2(t) = \frac{5 \sin t}{t} = 0 \Rightarrow \sin t = 0 \Rightarrow t = \pi$$

$$v_2(t) = 5 \left[\frac{t \cdot \cos t - \sin t \cdot 1}{t^2} \right] = 5 \left[\frac{t \cos t - \sin t}{t^2} \right]$$

Quotient Rule; simplify

$$a_2(t) = 5 \left[\frac{t^2(1 \cdot \cos t + t(-\sin t)) - \cos t - 2t(t \cos t - \sin t)}{t^4} \right]$$

Quotient Rule

$$= 5 \left[\frac{t^2 \cos t - t^3 \sin t - t^2 \cos t - 2t^2 \cos t + 2t \sin t}{t^4} \right]$$

Distribute

$$= 5 \left[\frac{-t^3 \sin t - 2t^2 \cos t + 2t \sin t}{t^4} \right]$$

Cancel terms

$$= 5 \left[\frac{-t^2 \sin t - 2t \cos t + 2 \sin t}{t^3} \right]$$

Cancel t

Velocity of Object B at time $t_b = \pi$:

$$v_2(\pi) = 5 \left[\frac{\pi \cos \pi - \sin \pi}{\pi^2} \right] = 5 \left[\frac{-\pi - 0}{\pi^2} \right] = -\frac{5}{\pi}$$

Speed of Object B at time $t_b = \pi$:

$$|v_2(\pi)| = \left| -\frac{5}{\pi} \right| = \frac{5}{\pi}$$

Acceleration of Object B at time $t_b = \pi$:

$$a_2(\pi) = 5 \left[\frac{-\pi^2 \sin \pi - 2\pi \cos \pi + 2 \sin \pi}{\pi^3} \right] = 5 \left[\frac{-2\pi(-1)}{\pi^3} \right] = \frac{10}{\pi^2}$$

1.6 1.7 1.8 *MNCsol.821 RAD Done

$$v_2(t) := \frac{d}{dt}(f_2(t))$$
$$v_2(t) \quad \left\{ \frac{5 \cdot \cos(t)}{t} - \frac{5 \cdot \sin(t)}{t^2}, t > 0 \right.$$
$$v_2(\pi) \quad \frac{-5}{\pi}$$

1.7 1.8 1.9 *MNCsol.821 RAD Done

$$a_2(t) := \frac{d^2}{dt^2}(f_2(t))$$
$$a_2(t) \quad \left\{ \left(\frac{10}{t^3} - \frac{5}{t} \right) \cdot \sin(t) - \frac{10 \cdot \cos(t)}{t^2}, t > 0 \right.$$
$$a_2(\pi) \quad \frac{10}{\pi^2}$$

- (c) Find the position of Object B at time t_a (the time found in part (a)). Are Objects A and B getting closer or are they getting farther apart at this time? Justify your answer.

Object B, time $t_a = \frac{\pi}{2}$:

$$y_2\left(\frac{\pi}{2}\right) = \frac{5 \sin(\pi/2)}{\pi/2} = \frac{10}{\pi}$$

$$v_2\left(\frac{\pi}{2}\right) = 5 \left[\frac{\frac{\pi}{2} \cos \frac{\pi}{2} - \sin \frac{\pi}{2}}{\left(\frac{\pi}{2}\right)^2} \right] = 5 \left[\frac{-1}{\frac{\pi^2}{4}} \right] = -\frac{20}{\pi^2} = -2.026$$

Therefore, Object B is at a position above 0 and moving downward.

Object A at time $t = \frac{\pi}{2}$:

$$y_1\left(\frac{\pi}{2}\right) = 0 \text{ and } v_1\left(\frac{\pi}{2}\right) = -5e^{-\pi/2} = -1.0394$$

Therefore, Object A is at position 0 and is also moving downward, but more slowly than Object B. So, the two objects are getting closer together at time $t_a = \frac{\pi}{2}$

Calculator window showing the calculation of $y_2\left(\frac{\pi}{2}\right)$ and $v_2\left(\frac{\pi}{2}\right)$. The results are $\frac{10}{\pi}$ and $-\frac{20}{\pi^2}$ respectively. A numerical approximation of $-\frac{20}{\pi^2}$ is shown as -2.02642 .

Calculator window showing the calculation of $y_1\left(\frac{\pi}{2}\right)$ and $v_1\left(\frac{\pi}{2}\right)$. The results are 0 and $-5 \cdot e^{-\pi/2}$ respectively. A numerical approximation of $-5 \cdot e^{-\pi/2}$ is shown as -1.0394 .

- (d) Find the position of Object A at time t_b (the time found in part (b)). Are Objects A and B getting closer or are they getting farther apart at this time? Justify your answer.

Object A, time $t_b = \pi$:

$$y_1(\pi) = 5e^{-\pi} \cos \pi = -5e^{-\pi}$$

$$v_1(\pi) = -5e^{-\pi}(\cos \pi + \sin \pi) = 5e^{-\pi} = 0.216$$

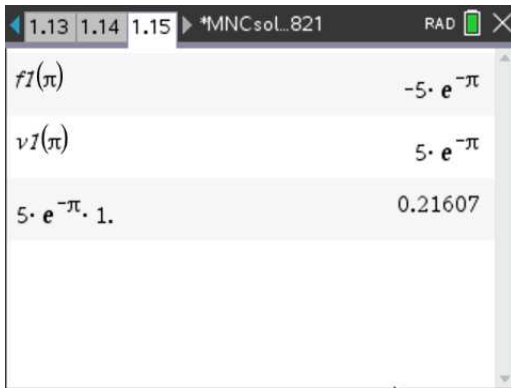
Therefore Object A is located at a position below 0 and is moving upward.

Object B at time $t_b = \pi$:

$$y_2(\pi) = 0 \text{ and } v_2(\pi) = 5 \left[\frac{\pi \cos \pi - \sin \pi}{\pi^2} \right] = 5 \left[\frac{-\pi}{\pi^2} \right] = -\frac{5}{\pi} = -1.592$$

Therefore Object B is at position 0 and moving downward.

So, the two objects are getting closer together at time $t_b = \pi$.



Calculator screenshot showing calculations for Object A at $t = \pi$. The window title is *MNCsol_821 and the mode is RAD. The calculations are:

$f1(\pi)$	$-5 \cdot e^{-\pi}$
$v1(\pi)$	$5 \cdot e^{-\pi}$
$5 \cdot e^{-\pi} \cdot 1.$	0.21607



Calculator screenshot showing calculations for Object B at $t = \pi$. The window title is *MNCsol_821 and the mode is RAD. The calculations are:

$v2(\pi)$	$-\frac{5}{\pi}$
$\frac{-5}{\pi} \cdot 1.$	-1.59155

(e) Over the time interval $0 \leq t \leq \pi$, find the average velocity of Object A and Object B.

Average velocity of Object A over $[0, \pi]$:

$$\frac{y_1(\pi) - y_1(0)}{\pi - 0} = \frac{-5e^{-\pi} - 5}{\pi} = -\frac{5}{\pi}e^{-\pi} = -1.660$$

Average velocity of Object B over $[0, \pi]$:

$$\frac{y_2(\pi) - y_2(0)}{\pi - 0} = \frac{0 - 5}{\pi} = -\frac{5}{\pi} = -1.592$$

A TI-84 Plus calculator screenshot showing the calculation of the average velocity for Object A. The display shows the formula $\frac{f1(\pi) - f1(0)}{\pi - 0}$ and the result -1.66033 . The input $-5 \cdot (e^{\pi+1}) \cdot e^{-\pi}$ is shown in the top right, and $\frac{-5 \cdot (e^{\pi+1}) \cdot e^{-\pi}}{\pi} \cdot 1.$ is shown in the bottom left.

A TI-84 Plus calculator screenshot showing the calculation of the average velocity for Object B. The display shows the formula $\frac{f2(\pi) - 5}{\pi - 0}$ and the result -1.59155 . The input $\frac{-5}{\pi} \cdot 1.$ is shown in the bottom left.

- (f) Which object traveled the greater total distance over the time interval $0 \leq t \leq 2\pi$? Show the computations that lead to your answer.

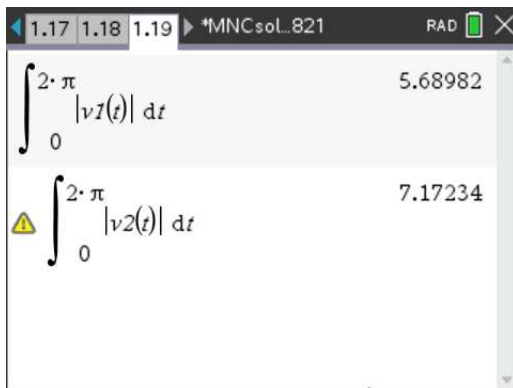
Total distance traveled by Object A over $[0, 2\pi]$:

$$\int_0^{2\pi} |v_1(t)| dt = 5.690$$

Total distance traveled by Object B over $[0, 2\pi]$:

$$\int_0^{2\pi} |v_2(t)| dt = 7.172$$

Therefore Object B traveled the greater distance over $[0, 2\pi]$.



(g) Find $\lim_{t \rightarrow \infty} (y_1(t) - y_2(t))$ or explain why the limit does not exist.

Consider $\lim_{t \rightarrow \infty} y_1(t)$: Use the Squeeze Theorem.

$$-e^{-t} \leq e^{-t} \cos t \leq e^{-t} \quad \text{for all } t > 0$$

$$\lim_{t \rightarrow \infty} -e^{-t} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} e^{-t} = 0$$

$$\text{Therefore } \lim_{t \rightarrow \infty} e^{-t} \cos t = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} 5e^{-t} \cos t = \lim_{t \rightarrow \infty} y_1(t) = 0$$

Similarly:

$$-\frac{1}{t} \leq \frac{\sin t}{t} \leq \frac{1}{t}$$

$$\lim_{t \rightarrow \infty} -\frac{1}{t} = \lim_{t \rightarrow \infty} \frac{1}{t} = 0$$

$$\text{Therefore } \lim_{t \rightarrow \infty} \frac{\sin t}{t} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{5 \sin t}{t} = \lim_{t \rightarrow \infty} y_2(t) = 0$$

$$\lim_{t \rightarrow \infty} (y_1(t) - y_2(t)) = 0 - 0 = 0$$



- (h) On the interval $0 \leq t \leq 2\pi$, at what time t are the two objects farthest apart? How far apart are they at this time?

Define $f(t) = y_2(t) - y_1(t) = \frac{5 \sin t}{t} - 5e^{-t} \cos t$

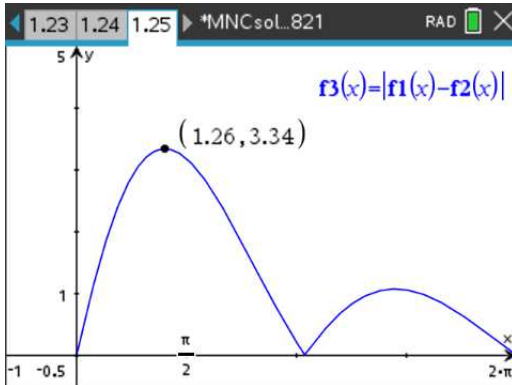
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1.19 1.20 1.21 *MNCsol..821 RAD
f(t):=f2(t)-f1(t) Done
g(t):=d/dt(f(t)) Done
z=zeros(g(t),t)|0<t<2*pi
  <t<2*pi and z={1.E-38,1.25975,4.55001}
  
```

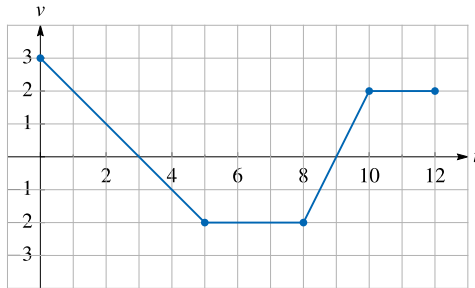
	A time	B distance	C	D
1	1.25975	3.34441		
2	4.55001	1.0759		
3	2.*π	0.009337		
4				
5				

The objects are farthest apart at time $t = 1.260$.

The objects are 3.344 units apart at that time.



2. The graph in the figure below shows the vertical velocity for an elevator as a function of time, where the velocity is measured in units of feet per second and time is measured in units of seconds, with $0 \leq t \leq 12$ seconds. The initial height, or position, of the elevator is $y(0) = 6$ feet above the ground.



- (a) Find the acceleration of the elevator at time $t = 2$ seconds. Indicate units of measure.

The slope of the velocity graph at time $t = 2$ seconds is -1 .

$$a(2) = v'(2) = -1 \text{ ft/s}^2$$

- (b) Is the elevator speeding up or slowing down at time $t = 4$ seconds? Explain your reasoning.

$$v(4) = -1 \text{ s} \quad \text{and} \quad a(4) = -1 \text{ ft/s}^2.$$

Since $v(4) < 0$ and $a(4) < 0$, the elevator is speeding up.

- (c) Find the average velocity of the elevator over the time interval $0 \leq t \leq 12$ seconds.

$$\frac{1}{12-0} \int_0^{12} v(t) dt = \frac{1}{12} \left(\frac{9}{2} - 2 - 6 - 1 + 1 + 4 \right) = \frac{1}{12} \cdot \frac{1}{2} = \frac{1}{24}$$

- (d) Find the time at which the elevator reaches its greatest height above the ground. What is that height?

$$\text{Height of the elevator: } y(t) = y(0) + \int_0^t v(x) dx = 6 + \int_0^t v(x) dx$$

$$y'(t) = v(t)$$

$$v(t) = 0: t = 3, 9$$

$v(t)$ DNE: none

t	$y(t)$
0	6
3	$6 + \frac{9}{2} = \frac{21}{2}$
9	$6 + \left(\frac{9}{2} - 9\right) = \frac{3}{2}$
12	$6 + \frac{1}{2} = \frac{13}{2}$

The maximum height of the elevator is $\frac{21}{2}$ feet, which occurs at time $t = 3$ seconds.

(e) Does the elevator ever go below ground level ($y = 0$)? Justify your answer.

The absolute minimum height of the elevator is $\frac{3}{2}$ feet.

Therefore the elevator never goes below ground level.

(f) Find the acceleration of the elevator when it is at its lowest level.

The elevator is at its lowest height at time $t = 9$.

The slope of the velocity graph is 2 at that time.

Therefore, $a(9) = v'(9) = 2 \text{ ft/s}^2$

(g) Find the height of the elevator at time $t = 12$ seconds.

$y(12) = \frac{13}{2}$ feet