Monday Night Calculus

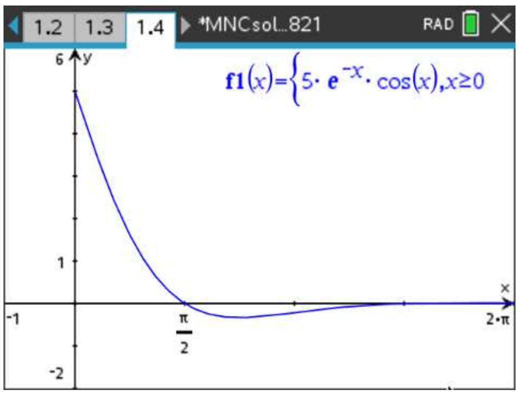
Rectilinear Motion

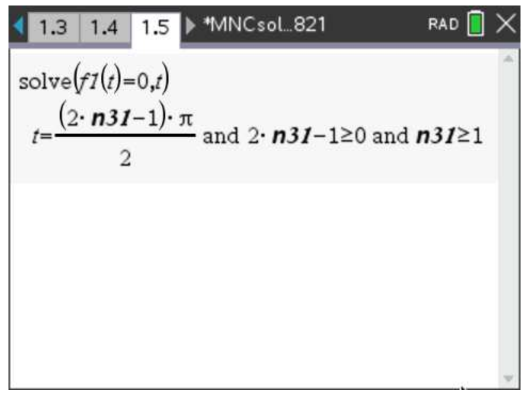
Exercises

1. Two objects oscillate along a vertical axis, starting at the same initial position $y = 5$ at time $t = 0$. The position of Object A at time t, $t \geq 0$, is given by $y_1(t) = 5e^{-t}\cos t$, and the position of Object B at time t, $t > 0$, is given by $y_2(t) = \dfrac{5\sin t}{t}$.

(a) Find the first time $t_a > 0$ at which Object A has position 0. What is Object A's velocity, speed, and acceleration at that time?

Consider a graph of Object A's position.





$$y_1(t) = 5e^{-t}\cos t = 0 \quad \Rightarrow \quad \cos t = 0 \quad \Rightarrow \quad t = \frac{\pi}{2}$$

$$v_1(t) = 5[-e^{-t}\cos t + e^{-t}(-\sin t)] \qquad \text{Product Rule}$$

$$= -5e^{-t}(\cos t + \sin t) \qquad \text{Simplify}$$

$$a_1(t) = -5[-e^{-t}(\cos t + \sin t) + e^{-t}(-\sin t + \cos t)] \qquad \text{Product Rule}$$

$$= -5e^{-t}[-\cos t - \sin t - \sin t + \cos t] \qquad \text{Factor}$$

$$= -5e^{-t}(-2\sin t) = 10e^{-t}\sin t \qquad \text{Simplify}$$

Velocity of Object A at time $t_a = \dfrac{\pi}{2}$:

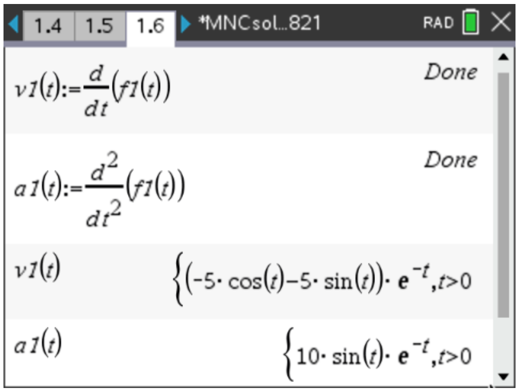
$$v_1\left(\frac{\pi}{2}\right) = -5e^{-\pi/2}\left[\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right] = -5e^{-\pi/2}(0+1) = -5e^{-\pi/2}$$

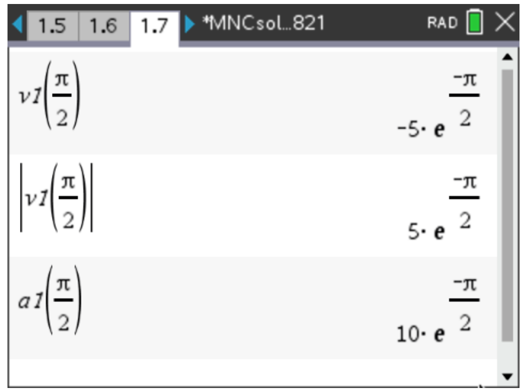
Speed of Object A at time $t_a = \dfrac{\pi}{2}$:

$$\left|v_1\left(\frac{\pi}{2}\right)\right| = \left|-5e^{-\pi/2}\right| = 5e^{-\pi/2}$$

Acceleration of Object A at time $t_a = \dfrac{\pi}{2}$:

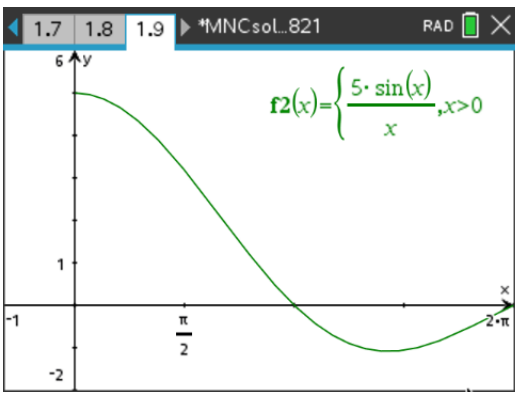
$$a_1\left(\frac{\pi}{2}\right) = 10e^{-\pi/2}$$

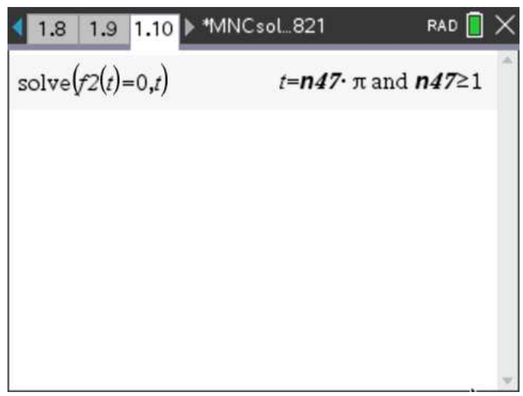




(b) Find the first time $t_b > 0$ at which Object B has position 0. What is Object B's velocity, speed, and acceleration at that time?

Consider a graph of Object B's position.





$$y_2(t) = \frac{5\sin t}{t} = 0 \quad\Rightarrow\quad \sin t = 0 \quad\Rightarrow\quad t = \pi$$

$$v_2(t) = 5\left[\frac{t\cdot\cos t - \sin t\cdot 1}{t^2}\right] = 5\left[\frac{t\cos t - \sin t}{t^2}\right]$$ Quotient Rule; simplify

$$a_2(t) = 5\left[\frac{t^2(1\cdot\cos t + t(-\sin t) - \cos t) - 2t(t\cos t - \sin t)}{t^4}\right]$$ Quotient Rule

$$= 5\left[\frac{t^2\cos t - t^3\sin t - t^2\cos t - 2t^2\cos t + 2t\sin t}{t^4}\right]$$ Distribute

$$= 5\left[\frac{-t^3\sin t - 2t^2\cos t + 2t\sin t}{t^4}\right]$$ Cancel terms

$$= 5\left[\frac{-t^2\sin t - 2t\cos t + 2\sin t}{t^3}\right]$$ Cancel t

Velocity of Object B at time $t_b = \pi$:

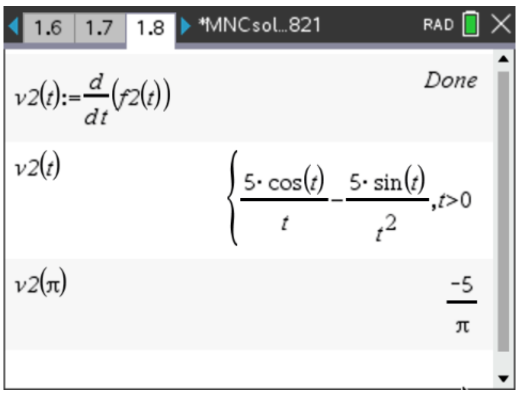
$$v_2(\pi) = 5\left[\frac{\pi\cos\pi - \sin\pi}{\pi^2}\right] = 5\left[\frac{-\pi - 0}{\pi^2}\right] = -\frac{5}{\pi}$$

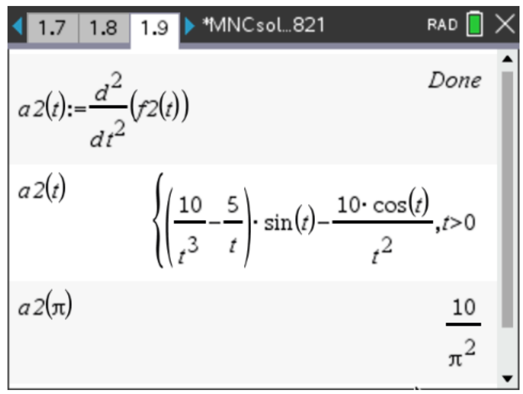
Speed of Object B at time $t_b = \pi$:

$$|v_2(\pi)| = \left|-\frac{5}{\pi}\right| = \frac{5}{\pi}$$

Acceleration of Object B at time $t_b = \pi$:

$$a_2(\pi) = 5\left[\frac{-\pi^2\sin\pi - 2\pi\cos\pi + 2\sin\pi}{\pi^3}\right] = 5\left[\frac{-2\pi(-1)}{\pi^3}\right] = \frac{10}{\pi^2}$$





(c) Find the position of Object B at time t_a (the time found in part (a)). Are Objects A and B getting closer or are they getting farther apart at this time? Justify your answer.

Object B, time $t_a = \frac{\pi}{2}$:

$$y_2\left(\frac{\pi}{2}\right) = \frac{5\sin(\pi/2)}{\pi/2} = \frac{10}{\pi}$$

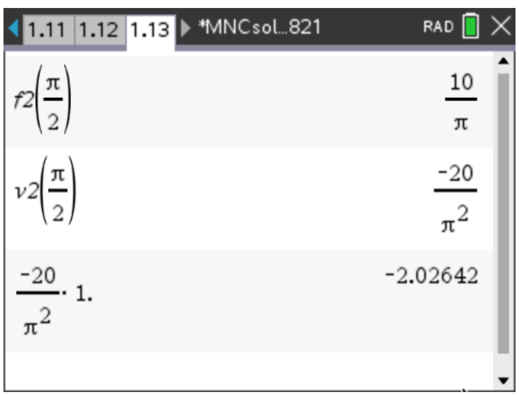
$$v_2\left(\frac{\pi}{2}\right) = 5\left[\frac{\frac{\pi}{2}\cos\frac{\pi}{2} - \sin\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)^2}\right] = 5\left[\frac{-1}{\frac{\pi^2}{4}}\right] = -\frac{20}{\pi^2} = -2.026$$

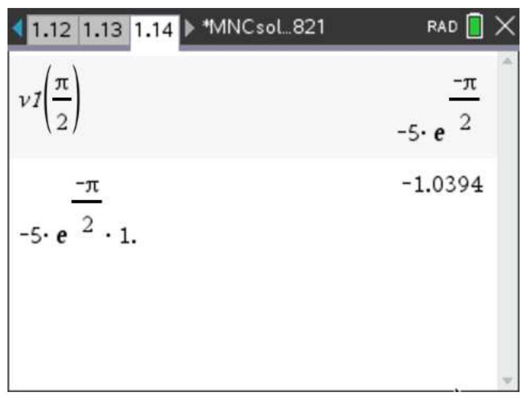
Therefore, Object B is at a position above 0 and moving downward.

Object A at time $t = \frac{\pi}{2}$:

$$y_1\left(\frac{\pi}{2}\right) = 0 \text{ and } v_1\left(\frac{\pi}{2}\right) = -5e^{-\pi/2} = -1.0394$$

Therefore, Object A is at position 0 and is also moving downward, but more slowly that Object B. So, the two objects are getting closer together at time $t_a = \frac{\pi}{2}$





(d) Find the position of Object A at time t_b (the time found in part (b)). Are Objects A and B getting closer or are they getting farther apart at this time? Justify your answer.

Object A, time $t_b = \pi$:

$y_1(\pi) = 5e^{-\pi}\cos\pi = -5e^{-\pi}$

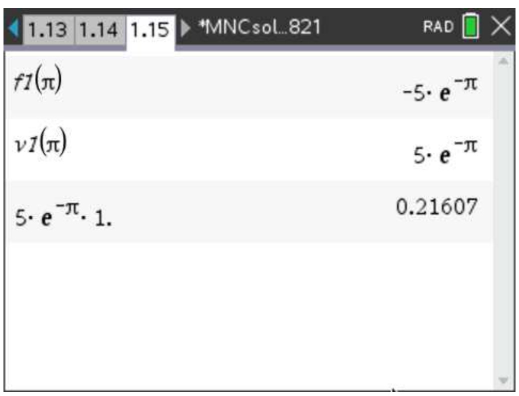
$v_1(\pi) = -5e^{-\pi}(\cos\pi + \sin\pi) = 5e^{-\pi} = 0.216$

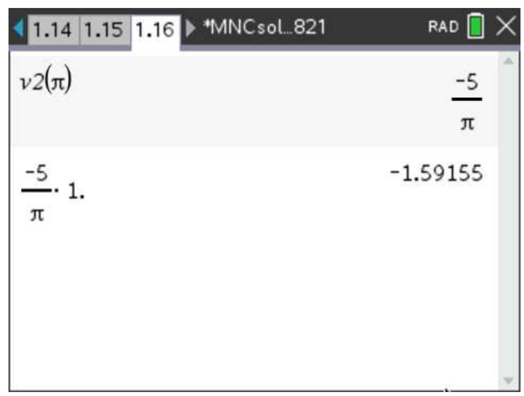
Therefore Object A is located at a position below 0 and is moving upward.

Object B at time $t_b = \pi$:

$$y_2(\pi) = 0 \text{ and } v_2(\pi) = 5\left[\frac{\pi\cos\pi - \sin\pi}{\pi^2}\right] = 5\left[\frac{-\pi}{\pi^2}\right] = -\frac{5}{\pi} = -1.592$$

Therefore Object B is at position 0 and moving downward.
So, the two objects are getting closer together at time $t_b = \pi$.





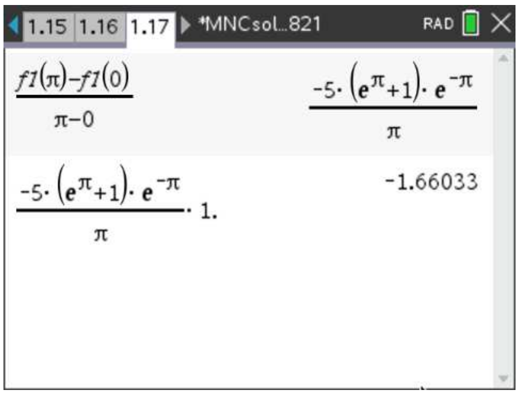
(e) Over the time interval $0 \le t \le \pi$, find the average velocity of Object A and Object B.

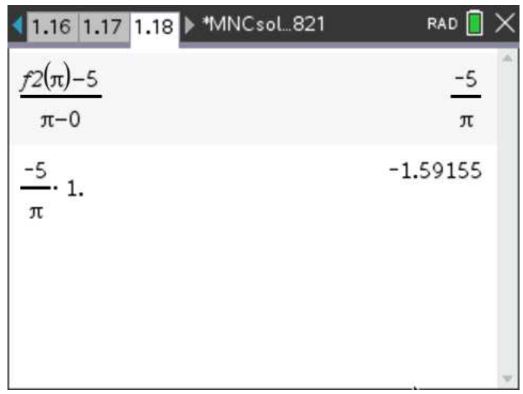
Average velocity of Object A over $[0, \pi]$:

$$\frac{y_1(\pi) - y_1(0)}{\pi - 0} = \frac{-5e^{-\pi} - 5}{\pi} = -\frac{5}{\pi}e^{-\pi} = -1.660$$

Average velocity of Object B over $[0, \pi]$:

$$\frac{y_2(\pi) - y_2(0)}{\pi - 0} = \frac{0 - 5}{\pi} = -\frac{5}{\pi} = -1.592$$





(f) Which object traveled the greater total distance over the time interval $0 \le t \le 2\pi$? Show the computations that lead to your answer.

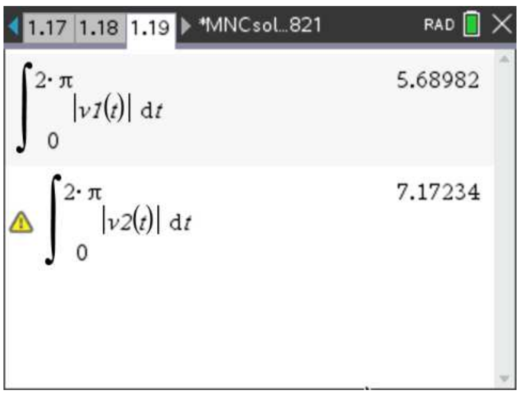
Total distance traveled by Object A over $[0, 2\pi]$:

$$\int_0^{2\pi} |v_1(t)|\, dt = 5.690$$

Total distance traveled by Object B over $[0, 2\pi]$:

$$\int_0^{2\pi} |v_2(t)|\, dt = 7.172$$

Therefore Object B traveled the greater distance over $[0, 2\pi]$.



(g) Find $\lim_{t\to\infty}(y_1(t) - y_2(t))$ or explain why the limit does not exist.

Consider $\lim_{t\to\infty} y_1(t)$: Use the Squeeze Theorem.

$-e^{-t} \le e^{-t}\cos t \le e^{-t}$ for all $t > 0$

$\lim_{t\to\infty} -e^{-t} = 0$ and $\lim_{t\to\infty} e^{-t} = 0$

Therefore $\lim_{t\to\infty} e^{-t}\cos t = 0$ and $\lim_{t\to\infty} 5e^{-t}\cos t = \lim_{t\to\infty} y_1(t) = 0$

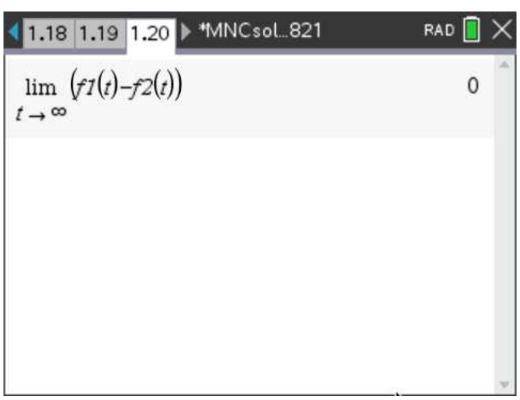
Similarly:

$$-\frac{1}{t} \le \frac{\sin t}{t} \le \frac{1}{t}$$

$$\lim_{t\to\infty} -\frac{1}{t} = \lim_{t\to\infty} \frac{1}{t} = 0$$

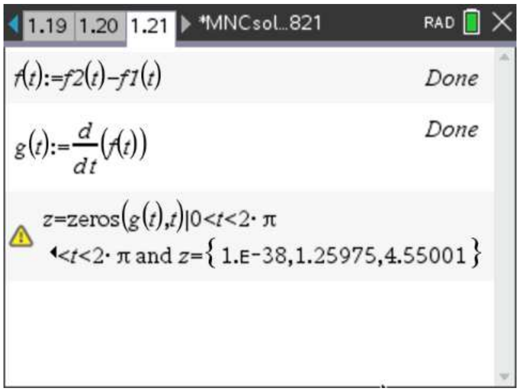
Therefore $\lim_{t\to\infty} \frac{\sin t}{t} = 0$ and $\lim_{t\to\infty} \frac{5\sin t}{t} = \lim_{t\to\infty} y_2(t) = 0$

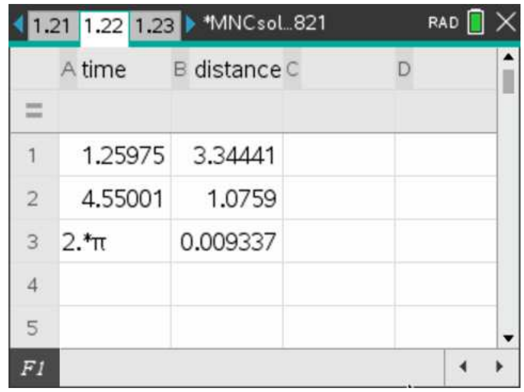
$$\lim_{t\to\infty}(y_1(t) - y_2(t)) = 0 - 0 = 0$$



(h) On the interval $0 \le t \le 2\pi$, at what time t are the two objects farthest apart? How far apart are they at this time?

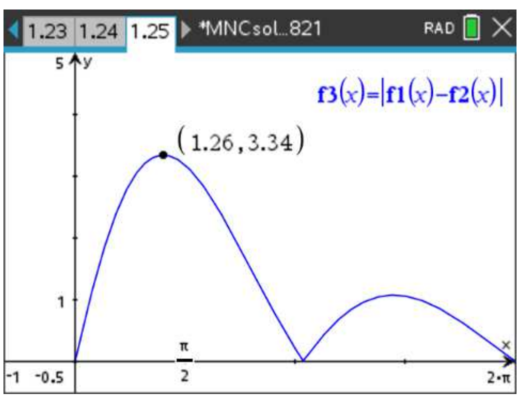
Define $f(t) = y_2(t) - y_1(t) = \dfrac{5\sin t}{t} - 5e^{-t}\cos t$



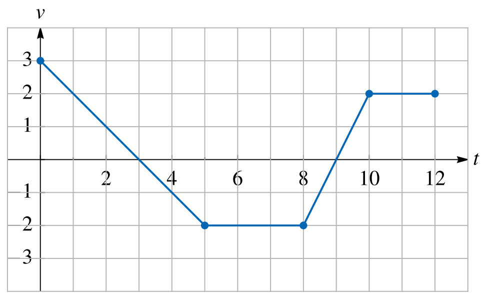


The objects are farthest apart at time $t = 1.260$.

The objects are 3.344 units apart at that time.



2. The graph in the figure below shows the vertical velocity for an elevator as a function of time, where the velocity is measured in units of feet per second and time is measured in units of seconds, with $0 \le t \le 12$ seconds. The initial height, or position, of the elevator is $y(0) = 6$ feet above the ground.



(a) Find the acceleration of the elevator at time $t = 2$ seconds. Indicate units of measure.

The slope of the velocity graph at time $t = 2$ seconds is -1.

$a(2) = v'(2) = -1$ ft/s^2

(b) Is the elevator speeding up or slowing down at time $t = 4$ seconds? Explain your reasoning.

$v(4) = -1$ s and $a(4) = -1$ ft/s^2.

Since $v(4) < 0$ and $a(4) < 0$, the elevator is speeding up.

(c) Find the average velocity of the elevator over the time interval $0 \le t \le 12$ seconds.

$$\frac{1}{12-0}\int_0^{12} v(t)\,dt = \frac{1}{12}\left(\frac{9}{2} - 2 - 6 - 1 + 1 + 4\right) = \frac{1}{12}\cdot\frac{1}{2} = \frac{1}{24}$$

(d) Find the time at which the elevator reaches its greatest height above the ground. What is that height?

Height of the elevator: $y(t) = y(0) + \int_0^t v(x)\,dx = 6 + \int_0^t v(x)\,dx$

$y'(t) = v(t)$

$v(t) = 0$: $t = 3,\ 9$

$v(t)$ DNE: none

t	$y(t)$
0	6
3	$6+\frac{9}{2}=\frac{21}{2}$
9	$6+\left(\frac{9}{2}-9\right)=\frac{3}{2}$
12	$6+\frac{1}{2}=\frac{13}{2}$

The maximum height of the elevator is $\frac{21}{2}$ feet, which occurs at time $t = 3$ seconds.

(e) Does the elevator ever go below ground level ($y = 0$)? Justify your answer.

The absolute minimum height of the elevator is $\frac{3}{2}$ feet.

Therefore the elevator never goes below ground level.

(f) Find the acceleration of the elevator when it is at its lowest level.

The elevator is at its lowest height at time $t = 9$.

The slope of the velocity graph is 2 at that time.

Therefore, $a(9) = v'(9) = 2\ \text{ft/s}^2$

(g) Find the height of the elevator at time $t = 12$ seconds.

$y(12) = \frac{13}{2}$ feet