CNEOS

Answers + Teacher Notes

7 8 9 10 **11** **12**

TI-Nspire™ Investigation Student 50 min

Introduction

CNEOS = Centre for Near Earth Object Studies, "Near Earth Objects" include comets and asteroids that have been attracted by the Earth's gravitational field. These objects are like time capsules; their composition remains relatively unchanged since the formation of the universe. The proximity and accessibility of NEOs make them more cost effective to access than planetary missions. The other significant interest in NEOs is the issue of whether or not they are likely to pose any impact risk with Earth. There are two programs that monitor potential threats:

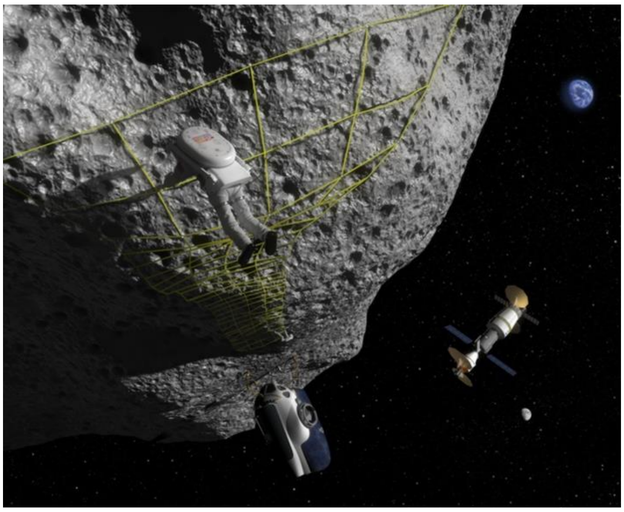


Image Courtesy of NASA/JPL-Caltech

Sentry Designed by CNEOS, this system monitors known NEOs for both short term and long term impact risk.

Scout This system monitors the short term impact risk for unconfirmed NEOs.

In this investigation you will explore a highly simplified mathematical model of these monitoring systems to determine:

- Shortest distance between a curve and a point
- Shortest distance between two curves

The shortest distance between two points on a flat surface is a straight line; space is three dimensional and curved! In this investigation we consider only the path of the object, in space these objects are moving, so too the reference frame from which the objects are observed. Despite the enormous differences between the complex Sentry system and the simplified problem presented here, the concepts developed in this investigation form a basis for understanding the combination of calculus and geometric concepts.

Teacher Notes:

To help theme the lesson, try including some live video stream from NASA's space station and a little applicable background music. NASA live stream available from YouTube: https://www.youtube.com/watch?v=qzMQza8xZCc

Another brilliant video that explains in very simple terms some of the mathematical complications involved in computing an asteroid's trajectory. Will 1999 RQ36 hit the earth in 2018? Watch the video to find out. https://www.youtube.com/watch?v=vizv4HlemnQ There are lots more great videos on the NASA Goddard channel.

A simple physical demonstration that helps students understand the geometry aspect of this problem is to have them standing in the middle of a room and walk towards a straight wall. Walk the shortest possible distance. The student's path will be perpendicular to the wall.

Author: P. Fox

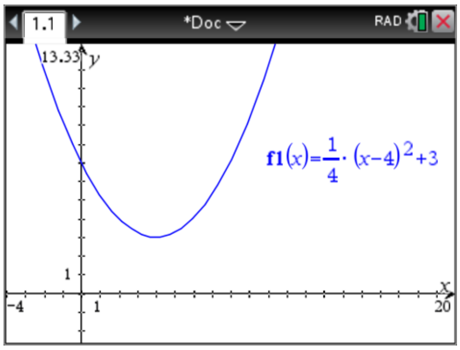
Shortest Distance between a Point and a Curve

A great way to solve a challenging problem is to start with an easier one.
Create a new TI-Nspire document and insert a Graph Application.
Graph the function:

$$f(x)=\frac{1}{4}(x-4)^2+3$$

Adjust the Window settings:

Menu: Window / Zoom > Zoom − Quadrant 1



Question: 1.

What is the distance between the origin (0, 0) and the y axis intercept?

Y intercept (0, 7), therefore distance = 7 units.

Question: 2.

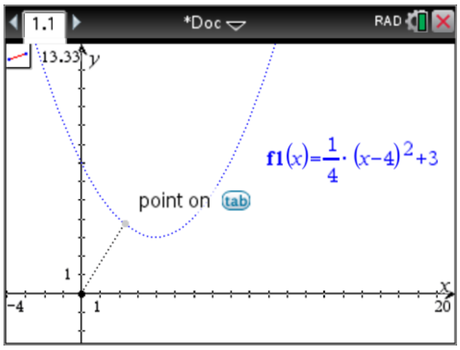
What is the distance between the origin (0, 0) and the turning point on the curve?

Turning point: (4, 3), using Pythagoras, distance = 5 units.

A line segment can be used to measure the straight line distance between the origin and a point on the curve.

Menu: Geometry > Points & Lines > Segment

When the line segment tool is active click on the origin (point of intersection) followed by a point on the curve (point on), as shown opposite.

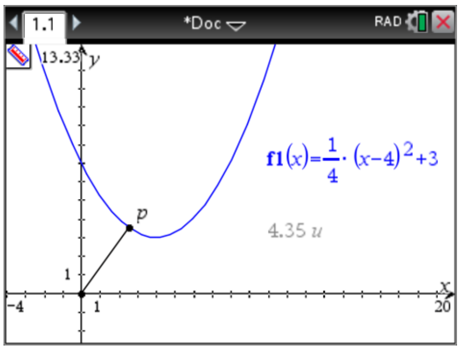


Note: The point on the curve will be referred to as point P.

The length of the line can be computed automatically.

Menu: Geometry > Measurement > Length

Click on the line segment then move the mouse to where you would like the measurement displayed and click again.
Press [ESC] to release the measurement tool. Place the mouse over point P on the graph, grab the point and move it along the graph.
Observe the physical change in length and corresponding measurement.



Question: 3.

Experiment with point P to find the shortest distance. According to your experimentation, what is the shortest distance between the origin and point P on the curve?

Shortest distance approximately 4.35 units, answers will vary slightly.

Question: 4.

The coordinates of point P corresponding to the shortest experimental distance can be obtained easily on the calculator. Place the mouse over point P and press [Ctrl] + [Menu], select Coordinates and Equations.

a. What are the coordinates of your point? (2.56, 3.52)

b. What is the gradient of this line segment? 1.375 (Approximate, answers will vary slightly.)

Question: 5.

Calculate the gradient of the curve at the experimental point P.

$$\frac{d\left(f_1(x)\right)}{dx} \mid x = 2.56$$

$$= -0.72$$

Question: 6.

Calculate the product of the gradient of the curve at point P (Q.5) and the gradient of the line segment (Q.4b). Remembering that these values are approximate, what does your answer suggest?

$$-0.72 \times 1.375 \approx -1$$

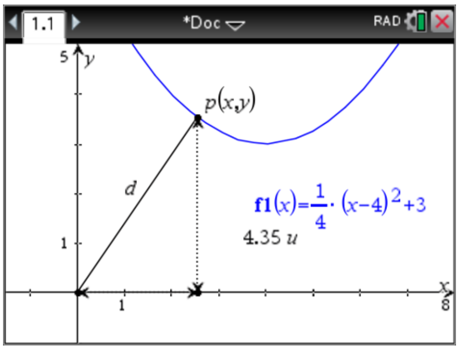
This suggests the line segment and curve are perpendicular, the line segment is normal to the curve.

The diagram opposite shows an enlarged and labelled version of the graph and line segment. The length of the line segment can be expressed as a function: $d(x)$ by using Pythagoras' theorem.

The base of the triangle has length x

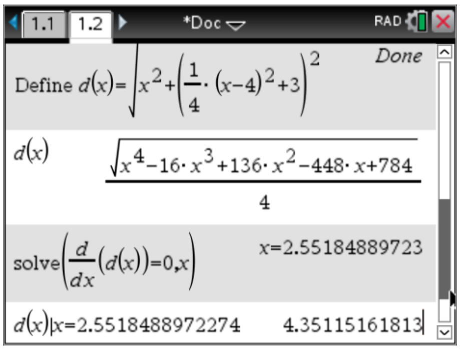
The height of the triangle has length y where $y = \frac{1}{4}(x-4)^2 + 3$

Define a function $d(x)$ using this information.



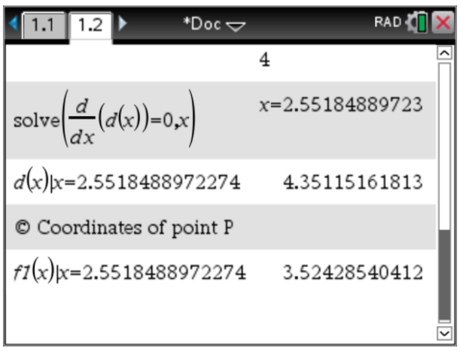
Question: 7.

Use calculus to determine the minimum distance between the origin and point P on the curve.



Question: 8.

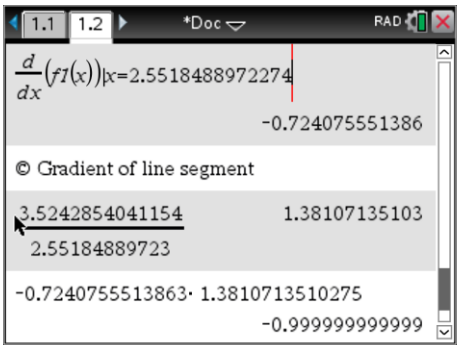
Determine the coordinates of point P corresponding to the minimum distance and calculate the gradient of the corresponding line segment.



Coordinates: (2.55, 3.52) Gradient of line segment: 1.38107135 …

Question: 9.

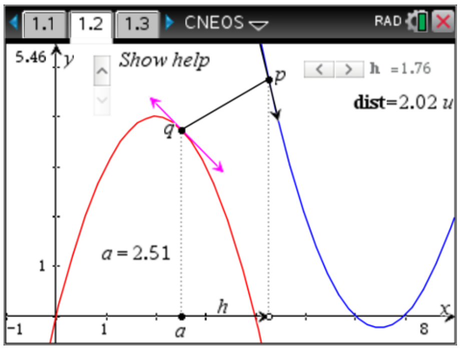
Use calculus to determine the gradient of $f(x)$ at point P, corresponding to the minimum distance.



Question: 10.

Calculate the product of the gradient (Q.9) and the line segment (Q.8). What does this result suggest about the relationship between the curve and the line segment? (See above) – Line segment is normal to the curve, or perpendicular to the tangent.

Shortest Distance between two Curves



Objects in space generally travel with a curved trajectory as they are influenced by the gravitational attraction of other bodies. Concepts developed in the previous questions will help explore this somewhat more challenging problem of finding the shortest distance between two curves.

Open the TI-Nspire file: CNEOS.

Navigate to page 1.2.

Point a can be moved along the x - axis and corresponds to point q on the function $f(x) = 4x - x^2$.

Point p is on the function $g(x) = x^2 - 13x + 42$ and is h units horizontally from q. Point p can be moved by adjusting the slider that controls h.

The length of the line segment $\overline{pq}$ is displayed as 'dist'.

A help option is available; when selected this will display the result of a calculation. The calculation method is not disclosed, but the value displayed may be helpful.

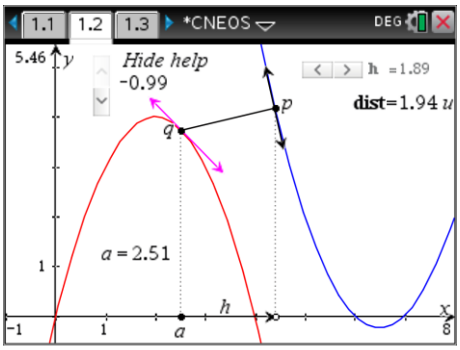
Question: 11.

Drag point a so that it has an x - coordinate of approximately 2.5. Use the slider to adjust the value of h until the distance between points p and q is a minimum. (Experimentally)

a. What is the distance between the two points?

Note: Help displays the active product of the gradient of the line segment and the tangent to the curve at point p.

Distance ≈ 1.94 units

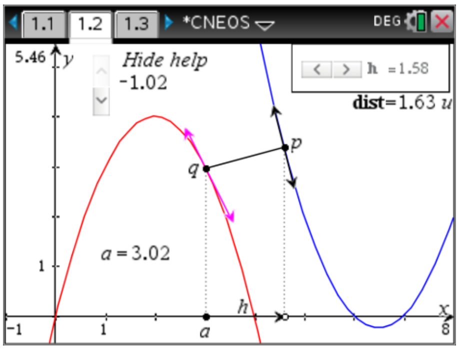


b. Explain how this problem relates to the problem explored previously.

Point q represents the fixed point (origin) in the first problem. The same conclusion is reached with respect to the line segment and the curve, it is normal to the curve.

Question: 12.

Drag point a so that it has an x - coordinate of approximately 3. Use the slider to adjust the value of h until the distance between points p and q is a minimum. (Experimentally)



a. What is the distance between the two points?

Distance ≈ 1.63 units

b. Based on your results from Q11a, Q12a and the visual representation of the problem, estimate the value of a for which the distance between the two curves will be a minimum.

Students may realise that the shortest distance will be found when the line segment is perpendicular with the tangent at q. The line segment is not quite perpendicular when a = 3 … Experimentally students will find that a will be approximately 3.4. (Answers will vary)

c. If points p and q are located such that the distance between $g(x)$ and $f(x)$ is a minimum, describe the relationship between the tangents to $g(x)$ and $f(x)$ at p and q.

The line segment connecting points p and q will be perpendicular to both tangents meaning that the tangents must be parallel, that is they will share the same gradient.

Teacher Notes:

This hypothesis is not proved here … teachers may elect to do a separate proof.

Question: 13.

Determine an expression for the gradient of $g(x)$ at any point x.

$$\frac{d(g(x))}{dx} = 4 - 2x$$

Question: 14.

Determine an expression for the gradient of $f(x)$ at any point $x + h$.

$$\frac{d(f(x+h))}{dx} = 2(x+h) - 13$$

Question: 15.

Using your conclusions from Q12.c, determine an expression for h in terms of x.

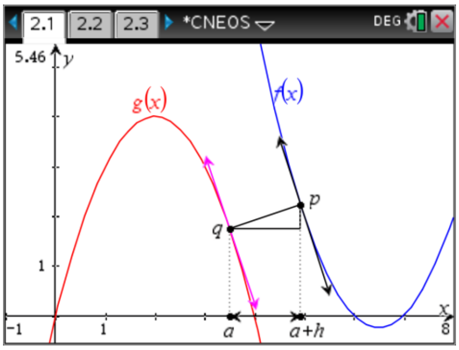
Answers from Q13 and Q14 must be equal from Q12 (parallel).

$$4 - 2x = 2(x+h) - 13$$

$$h = \frac{17 - 4x}{2}$$

Question: 16.

Use Pythagoras's theorem and the diagram opposite to determine a function $d(x)$ in terms of x.

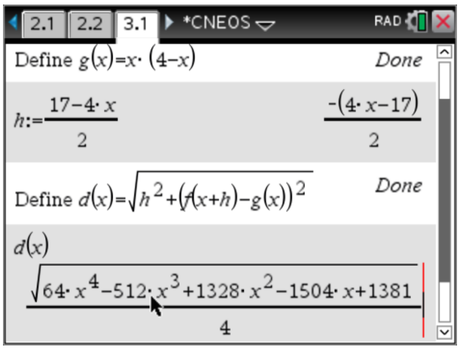


$$\text{distance } = \sqrt{h^2 + (f(x+h) - g(x))^2}$$

$$\text{given } h = \frac{17-4x}{2}$$

$$\text{distance} = \frac{\sqrt{64x^4 - 512x^3 + 1328x^2 - 1504x + 1381}}{4}$$

Comments: Students can use their CAS to determine this expression. They will however need to start a 'new problem' or use the Scratchpad as the *h* is already defined numerically in the diagram. Students can then define the distance as d(x) as shown below.



Question: 17.

Determine the minimum value of the distance function and the corresponding coordinates of points p and q.

$$\frac{d(d(x))}{x} = \frac{4(8x^3 - 48x^2 + 83x - 47)}{\sqrt{64x^4 - 512x^3 + 1328x^2 - 1504x + 1381}}$$

$$8x^3 - 48x^2 + 83x - 47 = 0$$

$$x \approx 3.535, \quad d \approx 1.5029$$

$$p = (4.965, 2.108), \quad q = (3.535, 1.642)$$

Comments: A worthwhile discussion / demonstration here relates to the potential simplification of the calculus. Determining the minimum distance requires the derivative to equal zero, the only requirement is therefore that the numerator equal zero. (Assuming the denominator is not equal to zero for the same x value.)

From the original definition of d(x) a by-hand short cut would be to 'ignore' the square-root component therefore simplifying the derivative process. "Why does this work?" "Will this strategy always work … for optimisation problems?"