

# Slope Fields & Differential Equations

## Answers

7 8 9 10 11 12



TI-Nspire



Investigation



Student



45 min

## Introduction

The purpose of this activity is to gain a conceptual understanding of slope fields and differential equations. The TI-Nspire file: "Slope Fields and DE's" includes interactive content required to answer the questions on this worksheet.

- Use the QR code or link to watch the concept video.
- Skim through this document before commencing the activity. Look at the pictures and read the annotations, this will 'prime' your brain in readiness for the content.
- Load the TI-Nspire file onto your calculator, the interactive content will help build understanding.
- Initial questions are designed to reinforce the visual representations on the calculator, the process of drawing them will help you remember.
- Subsequent questions will check your understanding. Some questions may seem repetitive; however they are designed to ensure you have understood each aspect of what is happening.



<http://bit.ly/SlopeFields>

## Problem 1

A function has the property that the gradient at any point  $(x_c, y_c)$  is equal to half the y coordinate at that point.

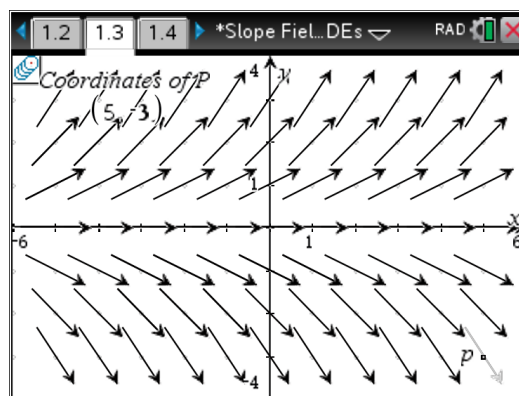
### Question: 1.

Complete the table below for the selected  $x$  and  $y$  coordinates.

Coordinates:	(2, 1)	(1, 2)	(-2, 2)	(-2, -2)	(2, -2)	(1, 0)	(0, 1)
Gradient:	0.5	1	1	-1	-1	0	0.5

### Question: 2.

Navigate to the graph application on page 1.3, drag point P around the screen. Use the screen image below to represent and reproduce the slope vector as it is moved around the screen.



**Question: 3.**

If point P is dragged parallel to the x axis, does the slope vector change? Explain your answer with regards to the visual representation and the problem description.

Slope vector doesn't change because the slope is dependent on y only, "gradient equal to half the y coordinate...", horizontal translation does not change y therefore slope vector does not change.

Isocline is derived from Greek words for "equal slope" (isos + klinein). For a first order differential equation, isoclines are points with the same numerical derivative. Re-read the original problem statement, move point P, observe the slope vector and the y coordinate.

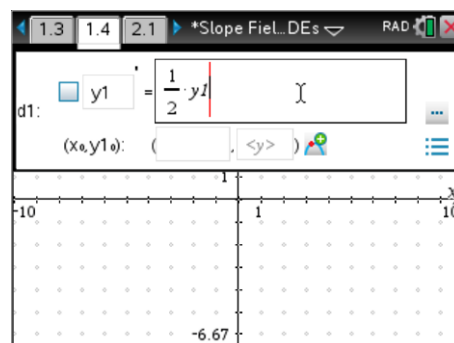
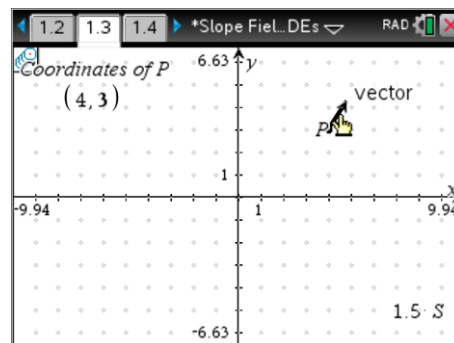
It is possible to 'retain' an image of the slope vectors using the **TRACE** feature.

From the **Trace** menu select **Geometry Trace**, move the mouse over the slope 'vector' then 'click'. Clicking on the vector means the vector will be reproduced (traced).

Move the mouse over Point P and grab it, (Ctrl + Click). Move it around the screen by gently swiping your finger across the touchpad. This will produce a semi-permanent record of the slope vector.

Navigate to page 1.4, use the menu to change the **Graph Entry** type to **Differential Equation** (Diff Eq).

For this space the standard entry is of the form:  $\frac{dy}{dx}$  or  $y'$



Each differential equation is assigned a number such as  $y_1$ ,  $y_2$  ... The equation must therefore include this numerical reference adjacent to each occurrence of the  $y$  variable in the equation. (Shown above)

**Question: 4.**

On page 1.4 enter the differential equation:  $\frac{dy}{dx} = \frac{y}{2}$ . Entered on the calculator as:  $y1/2$

a) Explain how the equation above relates to the problem statement.

Gradient  $\left(\frac{dy}{dx}\right)$  is equal to half the y coordinate  $\frac{y}{2}$  at that point, therefore  $\frac{dy}{dx} = \frac{y}{2}$

- b) Imagine a graph representing a solution to this differential equation. Is it possible for the graph to cross the  $x$  axis, explain.

It is not possible for the function to **cross** the  $x$  axis.

As  $y \rightarrow 0^+$ ,  $\frac{dy}{dx} \rightarrow 0^+$  (Graph is approaching  $x$  axis from 'above').

If  $y = 0$  then  $\frac{dy}{dx} = 0$  (Stationary point).

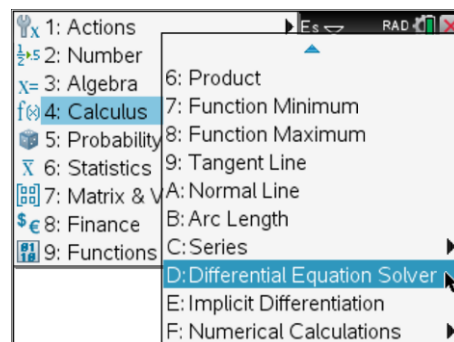
If the stationary point is a **point of inflection** the gradient of the function would need to be the same sign on either side of the point of inflection, however as  $y \rightarrow 0^-$ ,  $\frac{dy}{dx} \rightarrow 0^-$  and

$y \rightarrow 0^+$ ,  $\frac{dy}{dx} \rightarrow 0^+$  therefore the stationary point could not be a point of inflection.

A slope field is a graphical representation of the solution family to a differential equation. A family of vectors represent the slope of the graph at each point. The advantage of such a diagram is that it may be drawn without finding a specific or analytic solution.

The calculator is capable of resolving this differential equation. Navigate to page 1.5, from the **Calculus** menu select **Differential Equation Solver**. Enter the syntax shown below:

$$\text{deSolve}\left(y' = \frac{1}{2}y, x, y\right)$$



The first derivative symbol (prime) can be obtained from the punctuation key:

...

?	!	\$	°	'	%
"	:	;	-	\	

**Question: 5.**

What is the 'analytic' solution (according to the deSolve command) for the differential equation:

$$\frac{dy}{dx} = \frac{y}{2}$$

$$\text{deSolve}(y' = \frac{y}{2}, x, y) \quad y = c_1 e^{\frac{x}{2}}$$

**Question: 6.**

Transposing the differential equation and integrating both sides produces:

$$2 \int \frac{1}{y} dy = \int dx.$$

From this equation, determine an expression for  $y$  in terms of  $x$  and compare it with the one produced by the calculator command: deSolve. (Question 4)

$$2 \int \frac{1}{y} dy = \int dx$$

$$2 \ln(y) = x + c$$

$$y = e^{\frac{x+c}{2}}$$

$$y = ke^{\frac{x}{2}}$$

*Only one integration constant required*

*Since  $e^{\frac{x+c}{2}} = e^{\frac{x}{2}} \cdot e^{\frac{c}{2}}$  then  $e^{\frac{x+c}{2}} = k \cdot e^{\frac{x}{2}}$  where  $k = e^{\frac{c}{2}}$*

*Same as calculator answer.*

The analytic solution to this differential equation is a function that can be graphed on the same axis as the slope field.

Navigate to page 1.4, change the **Graph Entry** mode to **Function** and enter the function determined in Question 5.

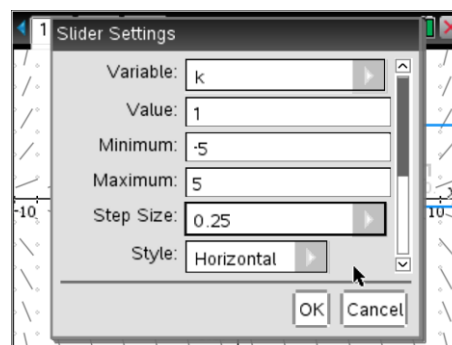
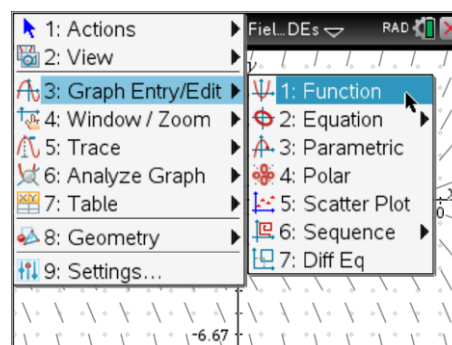
**Note:**

Enter the equation without the integration constant.

The integration constant will be dealt with next.

A slider can be used to accommodate the unknown integration 'constant' and further illustrate the general solution produced by the slope field.

From the **Action** menu, select **Slider** then match the settings shown opposite. The 'constant' assigned to the slider can then be used accordingly in the equation.



The general solution to a differential equation is shown by the slope field. A specific solution can be determined when a set of initial conditions are provided. The differential equation graphing tool uses Euler's method (default) to approximate the specific solution. The initial conditions help determine the integration constant in the analytic solution.

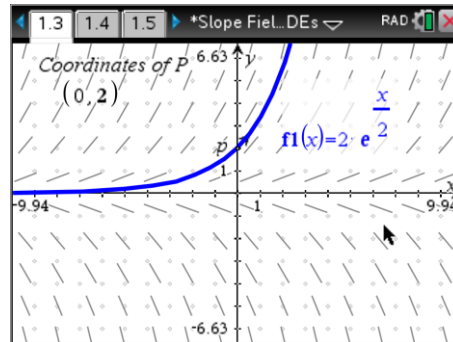
**Question: 7.**

Suppose the initial conditions for the differential equation are:  $x_0 = 0$  and  $y_0 = 2$ . Determine the integration constant and graph the corresponding analytic solution using the graph function.

$$y = ke^{\frac{x}{2}} \quad (0, 2)$$

$$2 = ke^{\frac{0}{2}}$$

$$y = 2e^{\frac{x}{2}}$$

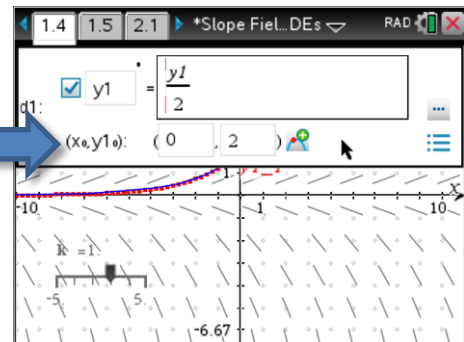


The calculator has the option to include initial conditions for each equation in the differential equation entry.

Change the **Graph Type** back to **Differential Equation**, navigate to Y1 then press **[Tab]** to navigate to the different settings for the differential equation. Enter the initial conditions. (0, 2)

**Note:**

The specific solution for the initial conditions, by default, is calculated using Euler's method. The accuracy of these calculations can be changed.

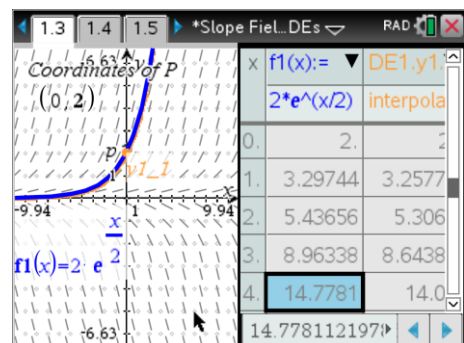
**Question: 8.**

How does the Euler approximation method 'perform' in relation to the graph determined by analytic means?

Euler approximation is very close to the analytic solution over the domain shown.

Press CTRL + T to view a table showing a numerical comparison of the analytic and approximate solution.

Note that as x increases the approximation is less accurate.



## Problem 2

A function has the property that the gradient at any point  $(x_c, y_c)$  is equal to the negative product of  $x_c$  and  $y_c$ .

### Question: 9.

Write the differential equation for the problem statement.

$$\frac{dy}{dx} = -xy$$

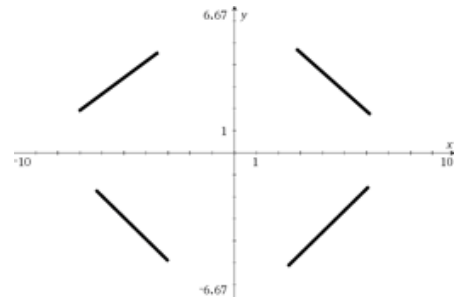
### Question: 10.

Study the differential equation and explain why all the slope vectors will have a negative gradient in the first quadrant.

In the first quadrant  $x > 0$  and  $y > 0$  therefore  $\frac{dy}{dx} = - \cdot + \cdot + = \text{negative}$

### Question: 11.

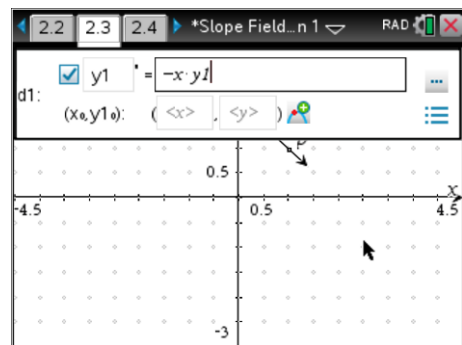
Consider the coordinates in each quadrant (ignore magnitude) and draw a single line in each quadrant to represent the typical direction of the slope vectors.



Navigate to page 2.2, select then move point P around the screen and confirm your answers to question 10.

Notice that point P is completely free to move, this flexibility is required later.

**Enter the differential equation**, drag point P around the slope field to compare the results.



### Question: 12.

Explain why the  $x$  and  $y$  axis are both isoclines.

Isoclines are lines of equal gradient. Given the differential equation  $\frac{dy}{dx} = -xy$ , so if  $x = 0$  or

$y = 0$  then  $\frac{dy}{dx} = 0$  therefore  $x$  and  $y$  axis have slope field lines with a gradient of 0. (Isoclines)

**Question: 13.**

Use the deSolve command to determine the general solution to this differential equation.

$$\text{deSolve}(y' = -xy, x, y) \quad y = ke^{\frac{-x^2}{2}}$$

**Question: 14.**

Solve the differential equation 'by hand'.

$$\begin{aligned} \frac{dy}{dx} &= -xy \\ \int \frac{1}{y} dy &= -\int x dx \\ \ln(y) &= \frac{-x^2}{2} + c \\ y &= ke^{\frac{-x^2}{2}} \end{aligned}$$

**Question: 15.**

Given initial conditions:  $x_0 = 0$  and  $y_0 = 2$ , determine the specific solution (by hand) to this differential equation. Check your answer using the deSolve command.

Initial conditions can be included in the deSolve command:

$$\text{deSolve}(y' = -xy \text{ and } y(0) = 2, x, y)$$

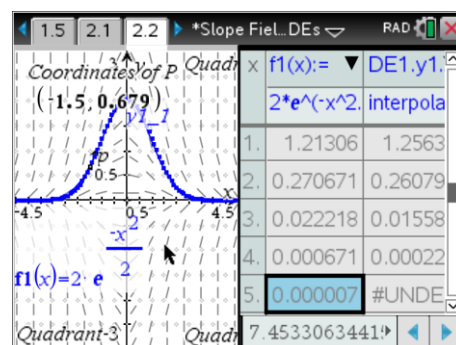
$$\begin{aligned} y &= ke^{\frac{-x^2}{2}} && (0, 2) \\ 2 &= ke^0 && \text{(Same answer as calculator)} \\ y &= 2e^{\frac{-x^2}{2}} \end{aligned}$$

**Question: 16.**

On page 2.2 graph the specific equation (Question 15) and compare this with the approximated version using Euler's method by inserting the initial conditions in Y1. (Diff Eq)  
Compare the accuracy of the Euler approximation to the specific solution obtained analytically.

Over the domain shown on the screen:  $-4 \leq x \leq 4$  the approximation is very accurate. To see the approximation on the table beyond these limits the window needs to be changed.

Further points 'appear' a lot different because the scientific notation is not completely visible in the table. So 2.53... looks different to 7.67, however these are actually  $2.53 \times 10^{-14}$  compared with  $7.67 \times 10^{-21}$  so the error is very small.



**Question: 17.**

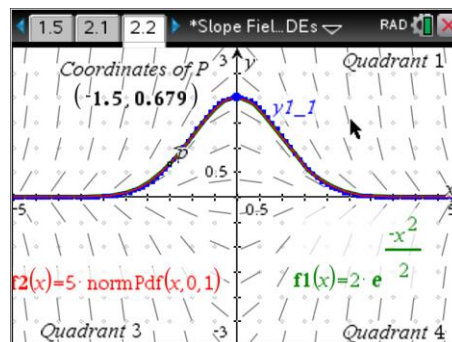
On page 2.2 graph the function:  $5 \cdot \text{normpdf}(x, 0, 1)$  and compare it with the specific solution to the differential equation.

**Hint:** The equation for the normal distribution is given by:  $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

The normal distribution equation is almost the same as the differential equation since  $\mu = 0$ ,  $\sigma = 1$  and

$\frac{1}{\sqrt{2\pi}} \approx 0.399$  then  $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$  becomes:

$$5 \cdot p(x) \approx 2e^{-\frac{x^2}{2}}$$

**Question: 18.**

Use the **differential equation**  $\frac{dy}{dx} = -xy$  and the second derivative test to determine the  $x$  coordinates for the points of inflection on the solution curve with initial conditions  $(0, 2)$ .



**Hint:** Use implicit differentiation.

$$\frac{dy}{dx} = -xy$$

$$\frac{d^2y}{dx^2} = y \frac{d(-x)}{dx} + (-x) \frac{d(y)}{dx}$$

$$\frac{d^2y}{dx^2} = -y - x \frac{dy}{dx} \quad \text{but } \frac{dy}{dx} = -xy$$

$$\frac{d^2y}{dx^2} = -y - x(-xy)$$

$$\frac{d^2y}{dx^2} = y(x^2 - 1) = 0 \quad \text{Second derivative test}$$

$$x = \pm 1 \quad \text{Since } y \neq 0$$

$$\text{Coordinates: } \left( \pm 1, \frac{2}{\sqrt{e}} \right)$$



**Question: 19.**

Use the specific solution to the differential equation with:  $x_0 = 0$ ,  $y_0 = 2$  and  $\frac{d^2y}{dx^2} = 0$  to verify the answer to Question 18.

$$y = 2e^{-\frac{x^2}{2}}$$

$$\frac{dy}{dx} = -2xe^{-\frac{x^2}{2}}$$

$$\frac{d^2y}{dx^2} = -2x \frac{d\left(e^{-\frac{x^2}{2}}\right)}{dx} + e^{-\frac{x^2}{2}} \frac{d(-2x)}{dx}$$

$$\frac{d^2y}{dx^2} = 2x^2 e^{-\frac{x^2}{2}} - 2e^{-\frac{x^2}{2}}$$

$$\frac{d^2y}{dx^2} = 2e^{-\frac{x^2}{2}} (x^2 - 1) = 0$$

$$x = \pm 1$$

*Second derivative test*

*As before...*

**Question: 20.**

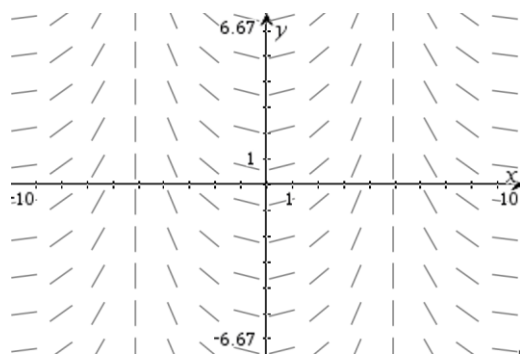
Each of the slope fields shown below is of the form:

Type 1:  $y' = f(x)g(y)$

Type 2:  $y' = f(x)$

Type 3:  $y' = g(y)$

In each case explain how you determined the form of the differential equation.



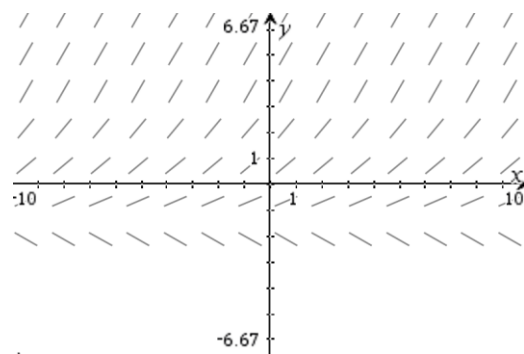
Slope vectors change from left to right and are therefore dependent on  $x$ .

Slope vectors do not change moving up and down therefore not dependent on  $y$ .

[Isoclines parallel to  $y$  axis]

Differential equation of the form:

$$y' = f(x)$$



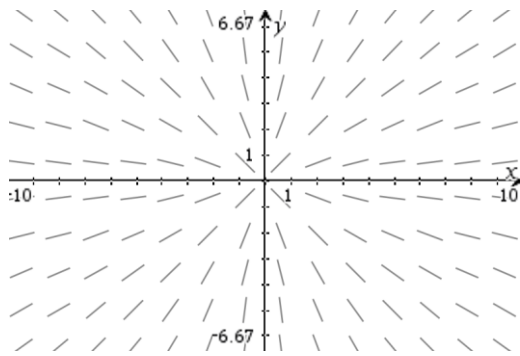
Slope vectors do not change from left to right and are therefore not dependent on  $x$ .

Slope vectors change moving up and down therefore dependent on  $y$ .

[Isoclines parallel to  $x$  axis]

Differential equation of the form:

$$y' = f(y)$$

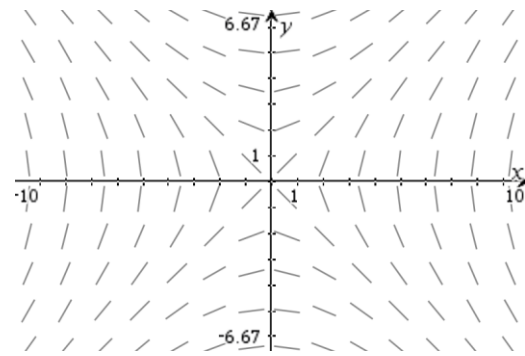


Slope vectors change from left to right and are therefore dependent on  $x$ .

Slope vectors change moving up and down therefore dependent on  $y$ .

Differential equation of the form:

$$y' = f(x)g(y)$$

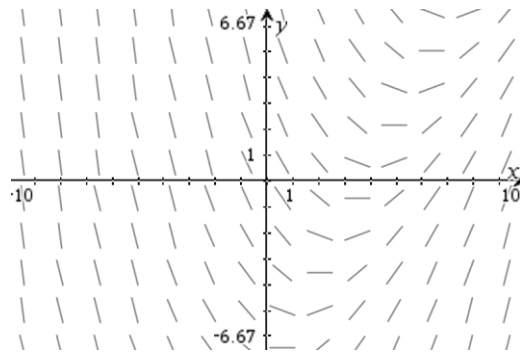


Slope vectors change from left to right and are therefore dependent on  $x$ .

Slope vectors change moving up and down therefore dependent on  $y$ .

Differential equation of the form:

$$y' = f(x)g(y)$$

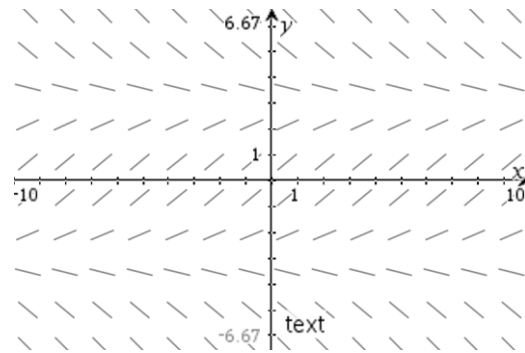


Slope vectors change from left to right and are therefore dependent on  $x$ .

Slope vectors change moving up and down therefore dependent on  $y$ .

Differential equation of the form:

$$y' = f(x)g(y)$$



Slope vectors do not change from left to right and are therefore not dependent on  $x$ .

[Isoclines parallel to  $x$  axis]

Slope vectors change moving up and down therefore dependent on  $y$ .

Differential equation of the form:

$$y' = f(y)$$