Factor & Remainder Theorem

Teachers Teaching with Technology™
Professional Development from Texas Instruments

Answers

7 8 9 10 **11** 12

TI-Nspire™ Investigation Student 50 min

Introduction

Division of whole numbers is a great way to start thinking about division of polynomials. When a number such as 36 is divided by 12 the result is 3. We can claim that 12 is a factor of 36 because there is no remainder. We can also subsequently claim that 3 is a factor of 36. Our result can be re-written as: 3 × 12 = 36.

Now consider 36 divided by 15; the result is 2 with 6 'left over'. The 2 is referred to as the quotient and the 6 'left over' is referred to as the remainder. We conclude that 15 is not a factor of 36 since the remainder is not zero. Our result can be re-written as: 15 × 2 + 6 = 36.

A polynomial $p(x)$ can be divided by another polynomial $g(x)$; if the remainder is zero then $g(x)$ must be a factor of $p(x)$. This can be written symbolically as: $p(x) \div g(x) = f(x)$, and the converse: $p(x) = f(x) \times g(x)$. The converse equation may be more familiar by consideration of a specific example:

$$f(x) = x+2 \;,\; g(x) = x+3 \text{ therefore } p(x) = (x+2)(x+3) \text{ or } p(x) = x^2 + 5x + 6$$

Now consider a polynomial $p(x)$ where $g(x)$ is not a factor. In this case the converse could be written as:

$$p(x) = f(x) \times g(x) + r(x) \text{ where } r(x) \text{ is the remainder function.}$$

This investigation explores polynomial division numerically, symbolically and graphically with a view to establishing a greater conceptual understanding.

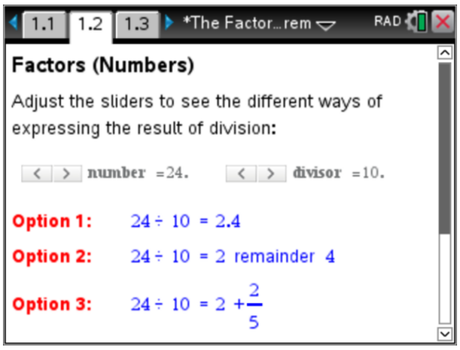
Instructions

Open the TI-nspire file "Factor and Remainder Theorem".

Navigate to page 1.2. Adjust the sliders to review the different ways that numerical division can be expressed, including the terminology:

- Quotient
- Remainder

You can scroll down to see the quotient and remainder expressed individually. Note the different ways of representing the numerical result.



Question: 1.

For each of the following write down the quotient and remainder:

a. 24 ÷ 10 = Quotient = 2 Remainder: 4

b. 30 ÷ 7 = Quotient = 4 Remainder: 2

c. 32 ÷ 6 = Quotient = 5 Remainder: 2

d. 36 ÷ 9 = Quotient = 4 Remainder: 0

Author: P. Fox

Question: 2.

For each of the results in Question 1, rewrite your answers as a product.

a) $2 \times 10 + 4 = 24$

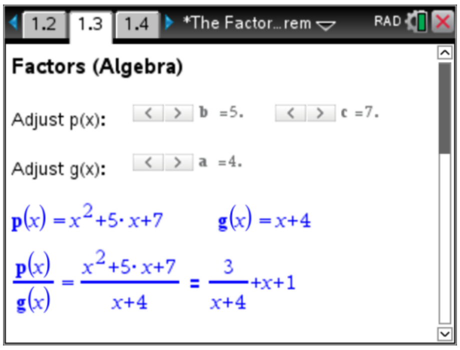
b) $4 \times 7 + 2 = 30$

c) $5 \times 6 + 2 = 32$

d) $4 \times 9 = 36$

Navigate to page 1.3.

Use the sliders to adjust the coefficients and constants in the two functions: $p(x)$ and $g(x)$.



Question: 3.

Use the sliders to set up the equations listed below. In each case state the quotient and remainder.

a. $\dfrac{x^2+6x+10}{x+2}$

Quotient = $x+4$ Remainder = 2

b. $\dfrac{x^2+5x+7}{x+3}$

Quotient = $x+2$ Remainder = 1

c. $\dfrac{x^2+4x+9}{x+3}$

Quotient = $x+1$ Remainder = 6

d. $\dfrac{x^2+6x+5}{x+5}$

Quotient = $x+1$ Remainder = 0

Question: 4.

Given: $\dfrac{x^2+7x+12}{x+2} = x+5+\dfrac{2}{x+2}$, relate the fractional component of the result: $\dfrac{2}{x+2}$ to the numerical results in Question 1 and the different options on page 1.2 of the TI-Nspire document.

The fractional component shows the remainder over the divisor: $x+2$ in much the same way as the remainder from: $30 \div 7$ is expressed as $\dfrac{2}{7}$, option 3 on page 1.2.

Question: 5.

Given: $\dfrac{x^2+8x+14}{x+2} = \dfrac{x^2+2x+6x+12+2}{x+2}$, express the numerator as three separate expressions and hence express $\dfrac{x^2+8x+14}{x+2}$ in quotient and remainder form.

$$\frac{x^2+2x+6x+12+2}{x+2} = \frac{x(x+2)}{x+2} + \frac{6(x+2)}{x+2} + \frac{2}{x+2}$$

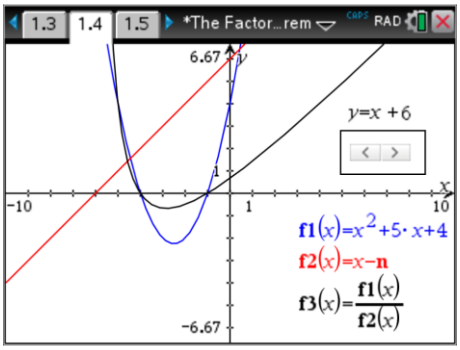
$$= x+6+\frac{2}{x+2}$$

Author: P. Fox

Navigate to page 1.4

The graphical representation of polynomial division helps to understand what is meant by "express the polynomial as a product of its linear factors".

In this graph $f_1(x)$ is divided by $f_2(x)$. Adjust the slider to change the expression for $f_2(x)$



Question: 6.

Adjust the slider until it is apparent that $f_1(x)$ is represented as the product of two linear factors:

a. Write down the linear factors of $f_1(x)$

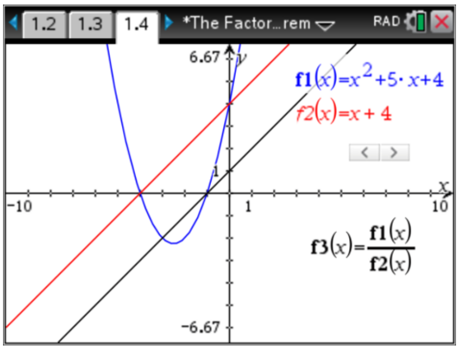
b. Where do the linear factors cross the x axis?

Linear factors: $x+4$ and $x+1$.

The linear factors cross at -1 and -4.

Teacher Notes:

The sudden change to two linear functions when a 'factor' is generated is visually impressive. The following questions are designed to make students realise that the product of the linear factors must be zero if either of the factors is zero.



Question: 7.

Given that $\dfrac{p(x)}{g(x)} = f(x)$ then it follows $p(x) = f(x)\cdot g(x)$. If $f(a)=0$ what will $p(a)$ equal?

If $f(a)=0$ then it follows that: $p(a)=0\cdot g(a)=0$ regardless of the value of $g(a)$.

Question: 8.

Discuss the relationship between the answers to Question 6(b) and Question 7 and the factor theorem that states:

If polynomial $p(x)$ has a root $x=a$ then $p(a)=0$ and $x-a$ is a factor of $p(x)$

Let $f(x)$ and $g(x)$ be linear factors of $p(x)$. It follows that the two linear factors must share x axis intercepts with the original function since if $f(a)=0$ then $p(a)=0$ and similarly if $g(a)=0$ (Question 7). If $f(x)$ is a linear function then it can be expressed in the form: $f(x)=mx+c$ or $f(x)=m(x-a)$ (Question 6) the latter clearly resulting in: $f(a)=0$.

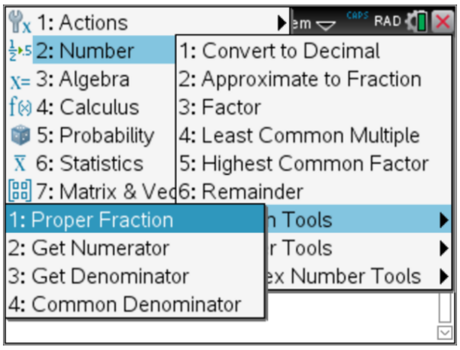
Author: P. Fox

Navigate to page 1.5

Page 1.5 is a calculator application. Use the menu to access the "Proper Fraction" command:

Number > Fraction Tools > Proper Fraction

A numerical fraction is 'improper' if the numerator is greater than the denominator. An algebraic fraction can be considered improper if the numerator is a higher degree polynomial than the denominator.



Question: 9.

Use the proper fraction command to re-write each of the following rational, algebraic fractions and identify the quotient and remainder for each.

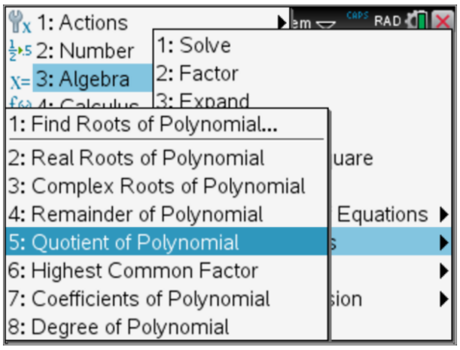
a. $\dfrac{x^2+6x+18}{x+5}$ $\quad x+1+\dfrac{13}{x+5}$ $\quad$ Quotient: $x+1$ $\quad$ Remainder: 13

b. $\dfrac{x^2-8x+12}{x+3}$ $\quad x-11+\dfrac{45}{x+3}$ $\quad$ Quotient: $x-11$ $\quad$ Remainder: 45

c. $\dfrac{x^3+4x^2+6x+11}{x+4}$ $\quad x^2+6-\dfrac{13}{x+4}$ $\quad$ Quotient: x^2+6 $\quad$ Remainder: -13

Additional polynomial tools allow you to use only the quotient or the remainder of polynomial division.

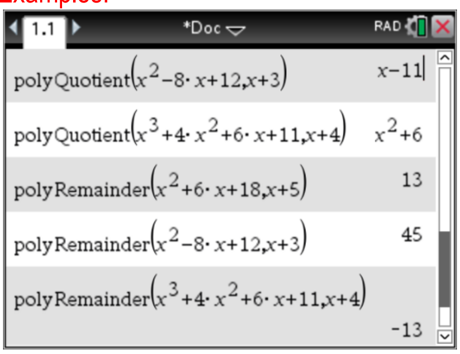
Algebra > Polynomial Tools > Quotient of Polynomial



Question: 10.

Use the **Remainder of Polynomial** and **Quotient of Polynomial** commands to check your answers to Question 9.

Examples:



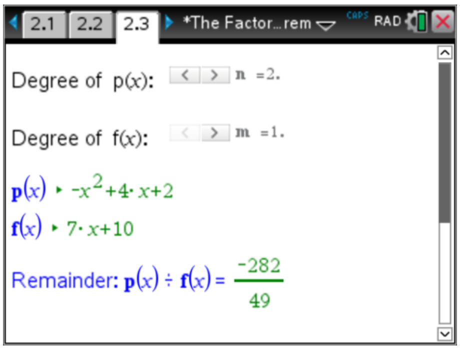
Author: P. Fox

Extension

Navigate to page 2.1.

In this section we are interested in the degree of the remainder when $p(x)$ is divided by $f(x)$.

Change the degree of each polynomial and observe the remainder, paying particular attention to the degree of the remainder.



Question: 11.

Explore the how the degree of $p(x)$ and $f(x)$ relate to the degree of the remainder. If $f(x)$ is a linear function (degree = 1), what will be the degree of the remainder?

Upon division: $p(x) \div f(x)$ results in a remainder with a degree at most one lower than the degree of $f(x)$.

Teacher Notes:

It is useful to compare this with the numerical approach. If a numerical quantity 'm' is divided by another numerical quantity 'n', then the remainder must be smaller than 'n'. For example 36 ÷ 10 must result in a remainder less than 10.

Question: 12.

If $p(x) = f(x)(x-a)+c$ and $c \neq 0$ then $(x-a)$ is not a factor of $p(x)$.

Determine an expression for $p(a)$ and discuss the significance of this result.

$p(a) = c$ since the product $f(x)(x-a) = 0$ when $x = a$. This is referred to as the remainder theorem. Furthermore since $(x-a)$ is linear then a must be a numerical quantity.

Author: P. Fox