# Linear Transformations

## Answers

7 8 9 10 11 **12**

TI-Nspire | Investigation | Student | 30 min

## Investigation

**Question: 1.**

The number appearing uppermost on a regular die is a discrete random variable. The uniform distribution of X can be represented as follows:

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $p(x)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

i) Determine the expected value for the distribution according to the rule: $\mu_x = E(X) = \sum_{i=1}^{n} x_i p(x_i)$

$$\mu_x = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6} = \frac{7}{2}$$

ii) Determine the variance and standard deviation for the distribution according to the rule using $\mu_x$ from the previous question:

$$\sigma_x^2 = Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$$

$$\sigma_x^2 = \left(1-\frac{7}{2}\right)^2 \times \frac{1}{6} + \left(2-\frac{7}{2}\right)^2 \ldots \left(6-\frac{7}{2}\right) \times \frac{1}{6} = \frac{35}{12}$$

$$\sigma == \frac{\sqrt{105}}{6}$$

iii) Determine the variance for the distribution according to the rule:

$$\sigma^2 = Var(X) = E(X^2) - E(X)^2 \text{ where } E(X^2) = \sum_{i=1}^{n} x_i^2 p(x_i)$$

$$\sigma_x^2 = \left(1^2\times\frac{1}{6} + 2^2\times\frac{1}{6} + \ldots 6^2\times\frac{1}{6}\right) - \left(1\times\frac{1}{6} + 2\times\frac{1}{6} \ldots 6\times\frac{1}{6}\right)^2 = \frac{35}{12}$$

**The mean: $\mu$ and variance: $\sigma^2$ of a random variable are called parameters as they are computed using theoretical probabilities. The mean: $\bar{x}$ and variance: $s_x^2$ of a sample are called statistics as they are computed using sample data.**



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**Question: 2.**

A non-standard die, shown opposite, is formed by increasing the number (dots) on each side of the die by 2. The uniform distribution of Y can be represented as follows:

| $y$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| $p(x)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

i) Without using any formulas, estimate the mean for this distribution and explain your reasoning.

Answers will vary – The simple application of a die is to promote the line of reasoning that "if every side of the die is simply increased by 2 then logical would suggest that average would increase by 2". This can easily be imagined by considering a game where a player moves forward by the amount appearing on the die.

ii) Without using any formulas, estimate the variance for this distribution and explain your reasoning.

Answers will vary – This concept is perhaps less understood. Ideally students will refer to the form of the distribution and reason that the variance will not change; however some students may incorrectly reason the variance will be smaller because by comparing effectively as a percentage of the expected value.

iii) Determine the expected value for the distribution according to the rule:

$$\mu_y = E(Y) = \sum_{i=1}^{n} y_i p(y_i)$$

$$\mu_y = 3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}+7\times\frac{1}{6}+8\times\frac{1}{6}=\frac{11}{2}$$

iv) Determine the variance for the distribution according to the rule using $\mu_Y$ from the previous question:

$$\sigma_y^2 = Var(Y) = \sum_{i=1}^{n} (y_i-\mu)^2 p(y_i)$$

$$\sigma_y^2 = \left(3-\frac{11}{2}\right)^2\times\frac{1}{6}+\left(2-\frac{11}{2}\right)^2\times\frac{1}{6}...\left(8-\frac{11}{2}\right)\times\frac{1}{6}=\frac{35}{12}$$

v) Determine the variance for the distribution according to the rule:

$$\sigma_y^2 = Var(Y) = E(Y^2)-E(Y)^2 \text{ where } E(Y^2)=\sum_{i=1}^{n} y_i^2 p(y_i)$$

$$\sigma_y^2 = \left(3^2\times\frac{1}{6}+4^2\times\frac{1}{6}...8^2\times\frac{1}{6}\right)-\left(3\times\frac{1}{6}+4\times\frac{1}{6}...8\times\frac{1}{6}\right)^2=\frac{35}{12}$$



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vi) Reflect and comment as applicable, on your estimates provided in parts (i) and (ii) with the computed results for parts (iii), (iv) and (v).

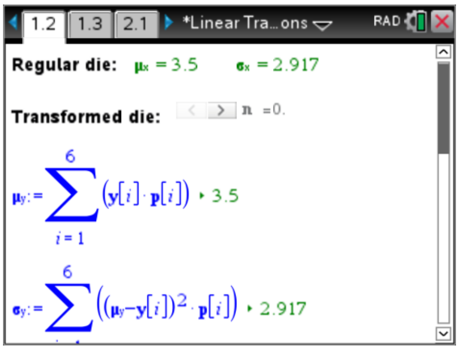
Answers will vary depending on whether students estimated the answers correctly. The most common scenario is students incorrectly estimating the variance, believing it too may change. The process of computing the variance by hand may help students identify that the actual computation is the same as the original distribution X even if they do not recognise the 'shape' of the distribution has not changed.

## Generalising the Result

The previous two distributions have been stored on the TI-Nspire calculator file, Linear Transformations. The random variable and respective probabilities have been stored as lists.

- x = $x$ values for Regular Die
- p = probabilities for Regular Die
- y = $y$ values for Transformed Die (The size of the *translation* can be controlled by a slider)

### Instructions:




Navigate to page 1.2, $\mu_x$ and $\sigma_x$ represent the mean and variance of the standard die. Use the slider to translate the distribution of X and therefore the subsequent distribution of Y.

Set the slider n = 2 to check your answers to the previous questions.

Note the similarity in the notation, where $y_i = y[i]$

**Question: 3.**

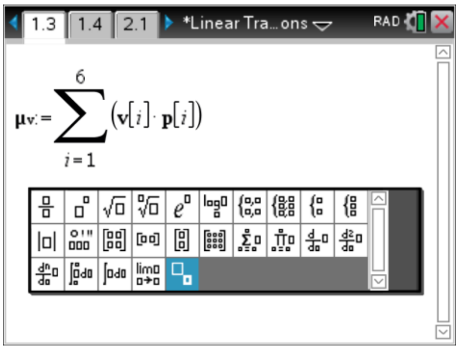
Use the slider to determine the mean and variance for a die with 5 extra spots on each side.

Mean of this distribution will be 8.5, the variance is unchanged.

**Question: 4.**

Describe the effect on the mean and variance of increasing the number of dots (value) on each side of the die.

Mean increases according to the number of dots added. No change to the variance.




Navigate to page 1.3, a calculator application. A third distribution, V has been included in this file. This distribution is a translation of the X distribution by '*a*' units.

The mean and variance can be determined using the same formulas outlined previously.

- The subscript label (v) for the mean can be accessed from the template menu. Once it has been assigned, the variable can be called up directly using the [VAR] key.
- Successive values for v and corresponding probabilities

stored as lists can be called upon notation shown opposite where $v_i = v[i]$ and $p_i = p[i]$.

Note that if $u_v$ is stored then it can be used directly in the calculation of the variance.

**Question: 5.**

Determine the mean and variance of distribution V, relate the results to your investigation so far and relate these to the general linear transformations of a random variable X:

$$E(X+b) = E(X)+b$$

$$Var(X+b) = Var(X)$$

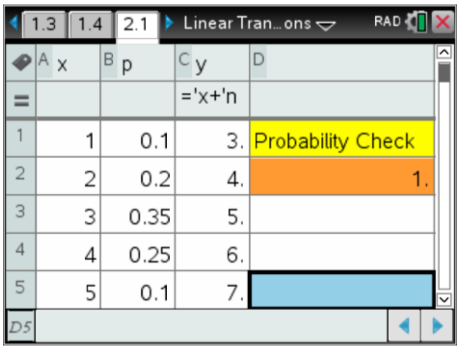
Mean of V distribution: $\mu_v = \frac{7}{2} + a$. This corresponds to the formula: $E(X+b) = E(X)+b$.

By practicing the by-hand approach in these relatively simple problems students may also see the algebraic formulation:

$$\begin{aligned}\mu_v &= (1+a)\times\frac{1}{6} + (2+a)\times\frac{1}{6}...(6+a)\times\frac{1}{6}\\ &= 1\times\frac{1}{6} + 2\times\frac{1}{6}...6\times\frac{1}{6} + a\times\frac{1}{6} + a\times\frac{1}{6}...a\times\frac{1}{6}\\ &= \mu_x + a\end{aligned}$$

Literally: If a common quantity (*a*) is added to each observed value in the distribution, the result is to increment the mean of the distribution by the same quantity (*a*).

## Non-Uniform Distributions:




So far all the distributions have been based on a uniform distribution that described the outcomes of rolling a die.

Navigate to page 2.1, the distribution is still discrete, but it is not uniform.

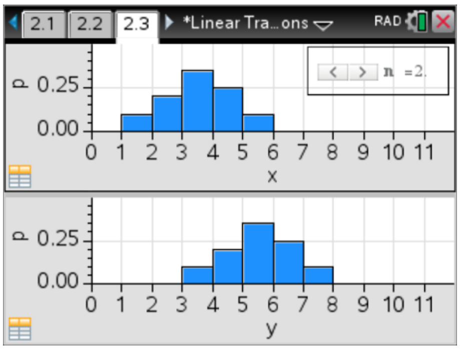
A probability check is located in cell D2.

The probabilities in column B (labelled P) can be changed, so too the outcomes $x$ in distribution X including the addition or removal of outcomes and their corresponding probabilities

The second distribution (Y) is a translation of X and will automatically update. **Do not directly adjust the outcomes for distribution Y.**

A pair of graphs on Page 2.3 show the original distribution X (top graph) and the translated distribution Y (bottom graph) where:

$$Y = X + b$$




Set the distribution to match the table below:

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p(x)$ | 0.1 | 0.2 | 0.35 | 0.25 | 0.1 |

**Question: 6.**

Determine the mean and variance of this distribution.

$$\mu_x = 1\times 0.1 + 2\times 0.2 + 3\times 0.35 + 4\times 0.25 + 5\times 0.1 = 3.05$$

**Question: 7.**

Adjust the slider (translation) so that n = 3. Describe the change that occurs to the mean of the translated distribution and explain how the visual shows that the variance does not change.

$$\mu_x = 3.05 + 3 = 6.05$$

The variance is unchanged; the 'shape' of the distribution does not change.

## Continuous Distribution

Continuous distributions can also be translated. For continuous distributions the formulas for mean and variance are as follows:

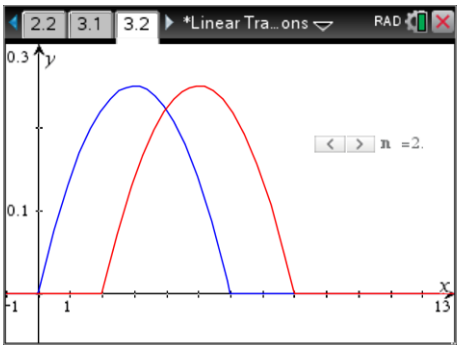
$$\mu = E(X) = \int_{-\infty}^{\infty} x.p(x)dx$$

$$\sigma^2 = Var(X) = \int_{-\infty}^{\infty} x^2.p(x)dx - \mu^2$$

### Instructions:

Navigate to page 3.1, a continuous random variable X has a distribution defined by:

$$f(x) = \begin{cases} \dfrac{x(6-x)}{36} & 0 \le x \le 6 \\ 0 & Else \end{cases}$$




A graph of the function is on page 3.2.



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**Question: 8.**

Verify that $f(x)$ is a probability density function.

$$\int_{-\infty}^{\infty} f(x)dx = 1 \text{ or } \int_{0}^{6} f(x)dx = 1$$

**Question: 9.**

Determine the mean and variance of this continuous distribution.

$$\mu_x = 3 \text{ and } \sigma_x^2 = \frac{9}{5}$$

**Question: 10.**

The distribution is translated (as shown above); determine the mean and variance of the translated distribution.

Students may apply reasoning from the discrete distribution: $\mu_x = 3 + 3 = 6$ and $\sigma_x^2 = \frac{9}{5}$ (as before). Alternatively students may use calculus: $\int_3^9 (x-3).f(x-3)dx = 6$ and similarly for the variance: $\sigma_x^2 = \int_3^9 (x-3)^2.f(x-3)dx - \mu_x^2 = \frac{9}{5}$.

**Question: 11.**

If $\Pr(\mu - a < x < \mu + a) = 0.95$ , determine the value of *a*.

Students will find that: $solve\left(\int_{\mu-a}^{\mu+a} f(x)dx = 0.95, a\right)$ does not work, even if $\mu$ is defined. The problem with this approach is that the probability distribution is defined as a piecewise function creating a potential conflict. Students can define: $g(x) = \frac{x(6-x)}{36}$ and then use the above approach.

$a \approx 2.4342$ which equates to: $\int_{0.5658}^{5.4342} f(x)dx \approx 0.95$

**Question: 12.**

Suppose Y is a continuous random variable such that Y = X + 3, determine the values of *a* if $\Pr(\mu - a < y < \mu + a) = 0.95$.

$a \approx 2.4342$ as the shape of the distribution has not changes by the translation, so *a* will remain the same distance from the mean.



Author: P.Fox