Bezier Curves

Teachers Teaching with Technology™
Professional Development from Texas Instruments

TI-Nspire™   Investigation   Student   120 min

Introduction

Pierre Bézier was a French engineer and well known for his work in computer aided design and manufacturing systems. Pierre developed what is now known as the Bézier curve. The curve was initially used to design smooth curves on motor vehicles, but soon became the standard in creating the smooth curves for fonts and also in computer animation. Bézier curves combine two important branches of mathematics, algebra and geometry.

Instructions – Building an envelope

Open a new TI-Nspire document, insert a **Graphs** application and save the file as "Bezier"

Change the view to include a dot grid.

> **menu** > **View** > **Grid** > **Dot Grid**

The dot grid will help keep numbers to integer values.

Create two line segments. The line segments need to meet at a common point. Example: segment 1:(-5, 5) - (-5, -3) and segment 2: (-5, -3) - (3, -3).

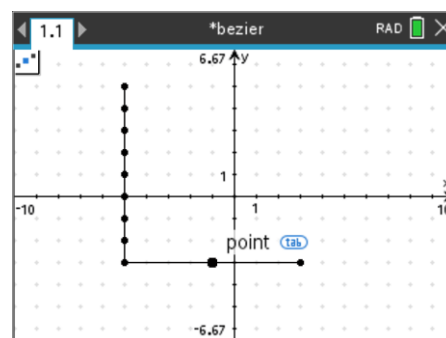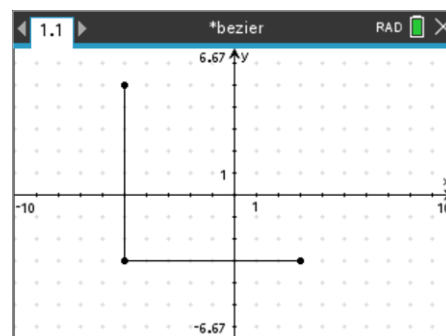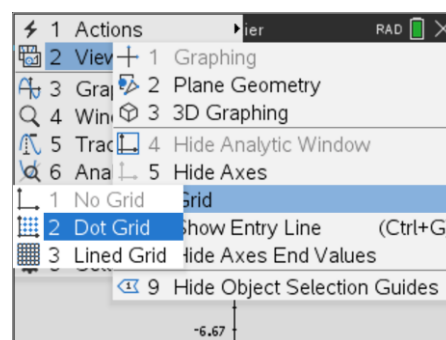> **menu** > **Geometry** > **Points and Lines** > **Segment**

The screen tip "Point On" appears when placing a point on the grid. Grid points are determined by the scale on each axis.

Press **esc** to release the segment drawing tool when finished.

A series of midpoints need to be constructed along each line. Midpoints will make it easy to ensure spacing is equal.

> **menu** > **Geometry** > **Construct** > **Midpoint**

Starting with the vertical segment, select each end of the segment. Place another midpoint from the middle of the line segment to one end. Continue placing midpoints so they are equally spaced along the line segment (shown opposite). Placing 7 midpoints will break the line segment into 8 equal pieces. Repeat this process for the horizonal line segment.
(Work in progress shown opposite)

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The final step is to join successive points along the respective line segments.

> menu > **Geometry** > **Points and Lines** > **Segment**

The first couple of line segments are shown opposite. Continue connecting the points. (8 line segments)
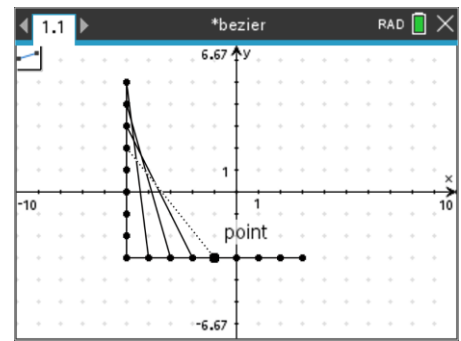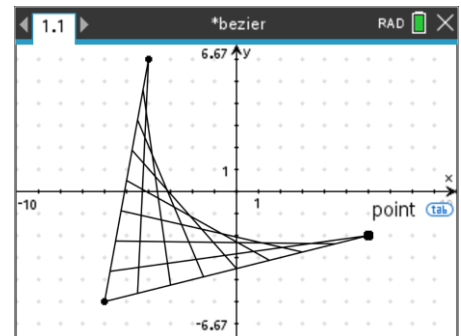
Once you have finished you can move the original line segments and watch the string of segments follow.

The image opposite has the midpoints hidden to make the image look a little cleaner. To hide the midpoints (do not delete):

> menu > **Actions** > **Show/Hide**

Click on each object you want to hide, then press esc when you're done.

The envelope formed by the successive straight lines is the curve that we are interested in, the more straight lines, the smoother the appearance of the curve. This particular curve can be modelled by an equation! The dynamic geometry environment can effectively model the curve using a 'locus'. A locus is a "set of all points, whose location satisfies or is determined by one or more specified conditions".

Instructions – Building a Locus

It's time to build another diagram.

Insert another **Graphs** application, add the grid and create a line segment starting at (2, 5) and finishing at (8, 5).
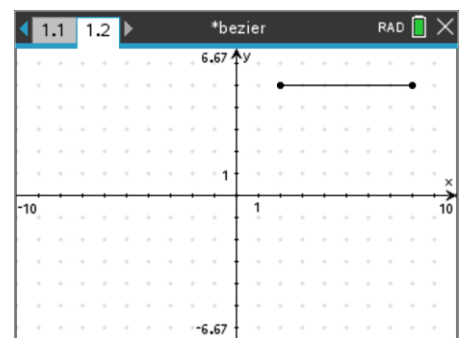
The **Points and Lines** menu can be used to create a point on the line segment, alternatively, press **P** on the keyboard and select **Point**.

Place a point on the line segment, this is the *control point*.

Measure the length of the line segment and also the distance from the start of the line segment to the new point. Place both of these measurements to the side.
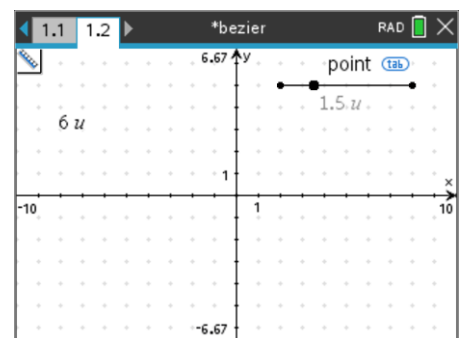
> menu > **Geometry** > **Measurement** > **Length**

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The next step is to determine the ratio between these two lengths. The ratio needs to generate values between 0 and 1. Placing a formula on the screen allows for dynamic calculations.

menu > **Actions** > **Text**

Type the expression: a ÷ b

The actions menu can now be used to calculate the expression.

menu > **Actions** > **Calculate**

Select the expression a ÷ b, then you will be prompted to select each of the variables in the expression (in alphabetical order). Click on the shorter measurement for 'a' and the longer measurement for 'b' and finally select the location of the resulting calculation.

Release the calculate tool and drag the point on the segment backwards and forwards.
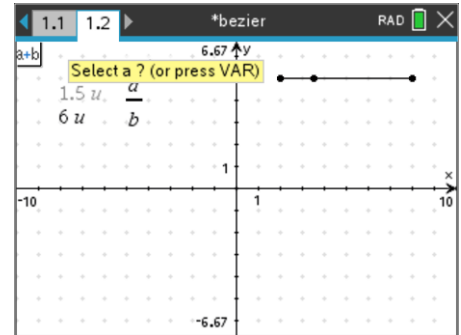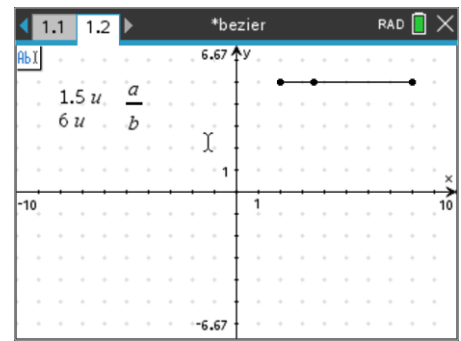
Question: 1

What values can be created as the point is moved along the line segment?
Answer: Values range between zero (far left) and one (far right).

Three points need to be created. For now, place one point in quadrant 1, another in quadrant 3 and the last in quadrant 4.

Placing the points on the grid is optional, however it will make calculations easier if the points are on the grid.
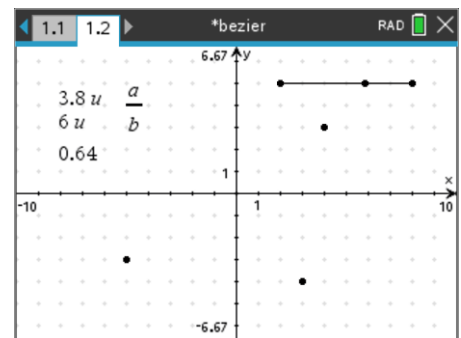
The next step is to use the dilation tool in the Geometry menu, this will help create the Bézier curve.

menu > **Geometry** > **Transformation** > **Dilation**

The dilation tool moves one point towards or away from another by a given factor. With the dilation tool active, select the point in the first quadrant, followed by the point in the fourth quadrant, then select the scale factor, the ratio calculated previously. (0.23 in the screen opposite).

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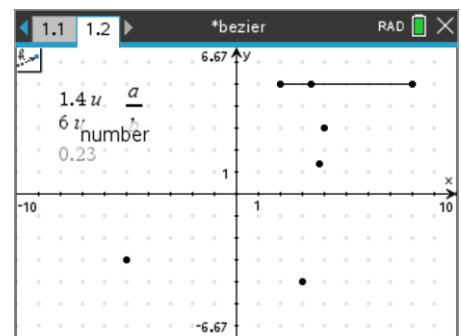
Repeat the dilation process by dilating the point in the fourth quadrant towards the point in the third quadrant using the same scale factor.

The two points created using the dilation tool have been highlighted in red and labelled to make this next instruction easier to follow.

Dilate point A towards point B using the same scale factor.
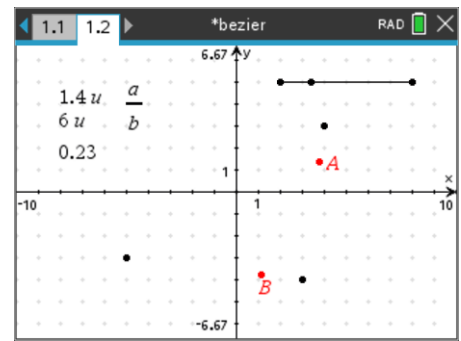
The dilations are now complete!

Move the control point on the first line segment, the point that drives the ratio, and observe how points A, B and B´ move.

Line segments and labels have been added to the diagram shown opposite to help referencing in the following questions. It is not necessary to add these to your diagram.
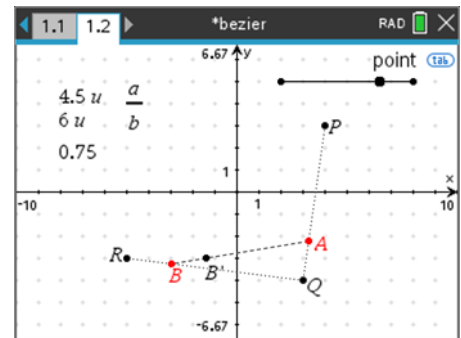
Question: 2
Describe the path of point A as the control point is moved.
Answer: Point A moves in a straight line starting at P (when ratio is 0) and finishing at Q when the ratio is 1. It's position along the line PQ is defined by the ratio. (dilation factor)

Question: 3
Describe the path of point B as the control point is moved.
Answer: Point B moves in a straight line starting at Q (when ratio is 0) and finishing at R when the ratio is 1. It's position along the line QR is defined by the ratio. (dilation factor)
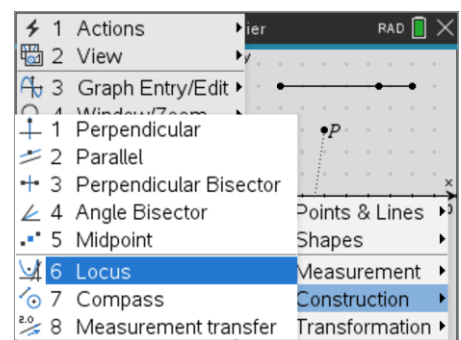
Question: 4
Describe the path of point B´ as the control point is moved.
Answer: Point B´ moves in a curve starting at A (when ratio is 0) and finishing at B when the ratio is 1.

The locus construction is finally ready.

menu > **Geometry** > **Construction** > **Locus**

Select the control point on the original line segment, followed by point B´.

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Question: 5

Explain how this locus is related to the envelope formed by the series of straight lines.

Answer: Each line in the family of straight lines connected points a set distance along the boundaries. For example; the first line in the family connected a point 1/8 of the distance AB to a point 1/8 of the distance BC where AB and BC are the original boundaries. This is the same as the dilation process in the creation of the locus.

Teacher Notes:

To help illustrate the connection between the locus and the original construction on Page 1.1, create a locus of the line segment connecting points A and B. (See opposite)
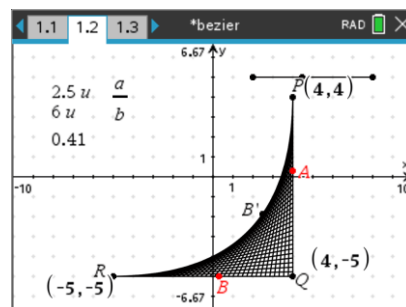
Question: 6

Move point R to (-2, 4), point Q to (0, -4) and P to (2, 4). Describe the shape of the locus (curve). Test your curve by entering an appropriate equation.

Answer: The locus forms the shape of a parabola.   Equation: $y = x^2$

Question: 7

Move point R to (-4, 4) and P to (4, 4), keep Q at (0, -4). Describe the shape of the locus (curve). Test your curve by entering an appropriate equation.

Answer: The locus forms the shape of a parabola.   Equation: $y = \dfrac{x^2}{4}$

Question: 8

With point R at (-4, 4) and P at (4, 4), move point Q to (0, -2). Describe the shape of the locus (curve). Determine the equation for the curve now.

Answer: The locus forms the shape of a parabola.   Equation: $y = \dfrac{3x^2}{16} + 1$

The equations to model the curve are relatively simple when points R and P are on the same horizontal line and Q is located on the axis of symmetry, midway between R and P. Bezier curves however can be moved anywhere. In this next section parametric equations will be used.

Instructions – Parametric Equations

To create a general equation for the Bezier curve the coordinates of points R, P and Q must be recorded or linked.

[menu] > **Actions** > **Coordinates and Equations**

Select each of the three points R, P and Q to display their coordinates. Click on the point and then again to place the coordinates in the desired location.

Note: Your coordinates may be different than those displayed.

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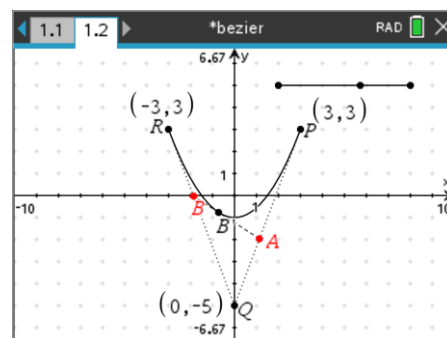
The coordinates can now be linked to variables. Move the mouse over the x coordinate (abscissa) of point R.

While the mouse is over the abscissa, press:

ctrl + menu

This activate the contextual menu, select **store** and then store the abscissa variable in "xr", signifying the x coordinate of point R.

When you press enter the abscissa will appear bold. Hovering the mouse over the value will initiate a pop-up message "Linked to xr".

Continue linking all the other coordinates:

R: (x_r, y_r), P(x_p, y_p), Q(x_q, y_q)

Once all the coordinates are linked. Change the graph type to parametric and enter the following equations:

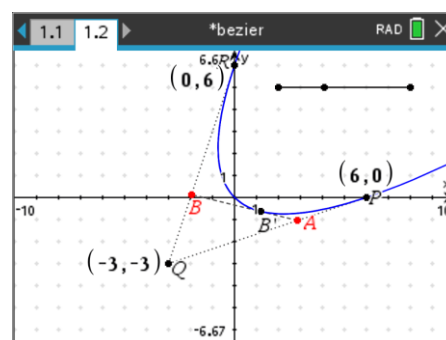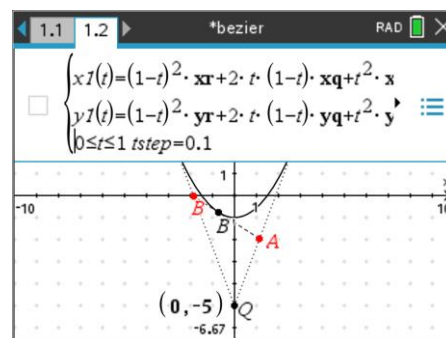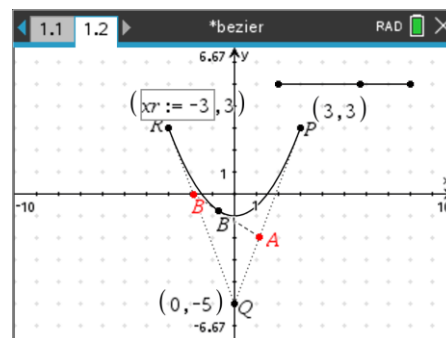$$x(t) = (1-t)^2 x_r + 2t(1-t)x_q + t^2 x_p$$

$$y(t) = (1-t)^2 y_r + 2t(1-t)y_q + t^2 y_p$$

Adjust the domain of t to vary between 0 and 1 and the t-step = 0.1

Adjust points R, P and Q, the parametric curve should now follow along with the Bezier curve.

Adjust points R, Q and P so they pass through (0, 6), (-3, -3) and (6, 0) respectively, as shown opposite.

Insert a Calculator application and recall the parametric equations.

Question: 9

a) What are the parametric equations for a Bezier curve passing with control points: (0, 6), (-3, -3) and (6, 0) (shown above)?

Answer: The equations can be recalled using x1(t) and y1(t) on the calculator application:
$$x(t) = 6t(2t-1) \quad \text{and} \quad y(t) = 6(t-1)(2t-1)$$

b) Determine the gradient in terms of t.

Answer: $\dfrac{dx}{dt} = 24t-6$ and $\dfrac{dy}{dt} = 24t-18$ therefore: $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{24t-18}{24t-6}$ or $\dfrac{dy}{dx} = 1 - \dfrac{2}{4t-1}$

c) Determine the Cartesian equation for your Bezier curve in the general form of a conic section:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Answer: $x^2 - 2xy + y^2 - 6x - 6y = 0$

d) Determine the gradient of the Bezier curve in terms of x and y.

Answer: $2x - 2x\dfrac{dy}{dx} - 2y + 2y\dfrac{dy}{dx} - 6 - 6\dfrac{dy}{dx} = 0$ therefore $\dfrac{dy}{dx} = \dfrac{x-y-3}{x-y+3}$

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e) Determine the equations the tangents at (6, 0) and (0 6). Comment on the point where these tangents intersect.

Answer: Tangent at (6, 0) has a gradient of $\dfrac{6-0-3}{6-0+3}=\dfrac{1}{3}$ therefore tangent through (6, 0): $y=\dfrac{1}{3}(x-6)$

Tangent at (0, 6) has a gradient of: $\dfrac{0-6-3}{0-6+3}=3$ therefore tangent through (0, 6): $y=3x+6$

These tangents meet at the point: (-3, -3) which is point Q!

f) Determine the coordinates for the turning point on the curve.
Answer: Techniques will vary; however students should see by symmetry (y = x) that the turning point falls on the origin: (0, 0). Students may choose to confirm this by showing the gradient at (0, 0) is equal to -1, at right angles to the axis of symmetry.

Extension

Question: 10

Points P, and R remain as (6, 0) and (0, 6) but Q is moved to (0, -6).

a) Write down the parametric equations.

Answer: $x(t)=6(t-1)^2$ and $y(t)=6t(3t-2)$

b) Determine the Cartesian equation in the general form: $ax^2+bxy+cy^2+dx+ey+f=0$.

Answer: $9x^2-6xy-60x+y^2-12y+36=0$.

c) From the general form if $b^2-4ac=0$ then the graph is a parabola. Show that the graph is a parabola.
Answer: $b=-6,\ a=9\ and\ c=1\ \ \therefore 36-4\times 9\times 1=0$.

d) Determine the gradient of the curve in terms of t.

Answer: $\dfrac{dx}{dt}=12(t-1)$ and $\dfrac{dy}{dt}=12(3t-1)$ therefore: $\dfrac{dy}{dx}=\dfrac{dy}{dt}\times\dfrac{dt}{dx}=\dfrac{12(3t-1)}{12(t-1)}=\dfrac{3t-1}{t-1}$

Teacher Notes: Students may also see that the derivative can be written in terms of x given x(t).

e) Use implicit differentiation to determine the gradient of the curve.

Answer: $\dfrac{dy}{dx}=\dfrac{9x-3y-30}{3x-y+6}$

Teacher Notes: Students are given the instructions to use implicit differentiation here as the gradient can also be determined from the previous answer as: $\dfrac{dy}{dx}=3-2\sqrt{\dfrac{6}{x}}$

f) From part(c) we know the curve is a parabola, however it has been rotated by some angle θ. The angle of rotation can be determined using: $x=x'\cos(\theta)-y'\sin(\theta)$ and $y=x'\sin(\theta)+y'\sin(\theta)$. Once these substitutions have been made, determine the value of θ such that the component $x'y'=0$

Note: $0\le\theta\le\dfrac{\pi}{2}$

Answer: $\theta=\operatorname{Tan}^{-1}(3)$

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g) User your answer from the previous question to determine the coordinates of the turning point.

Answer: From the previous question, the axis of symmetry has a gradient of 3, therefore the tangent through the turning point will have a gradient of -1/3.

$$\therefore solve\left(\frac{dy}{dx} = \frac{9x - 3y - 30}{3x - y + 6} = -\frac{1}{3}, y \right)$$ and substitute the result into the original equation to get x.

Solution: $x = \frac{54}{25}$ *and* $y = -\frac{48}{25}$. Note that the solution is the same using $\frac{dy}{dx} = 3 - 2\sqrt{\frac{6}{x}} = -\frac{1}{3}$

h) Use the Analyse Graph option and select Analyse Conics to determine the equation for the axis of symmetry for the Bezier curve. (Requires Cartesian equation).

Answer: $y = 3x - \frac{42}{5}$

i) Determine the location of the "turning point" on the Bezier curve using the point of intersection tool.

Answer: The calculator only produces and approximate value, which is approximately equal to part(g). Students can determine this value using 'point of intersection' between the axis of symmetry and the original curve. Students may also insert the directrix and focus and find the midpoint, this finds the 'exact' coordinates for the turning point.

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